Cosmic Large-scale Structure Formations

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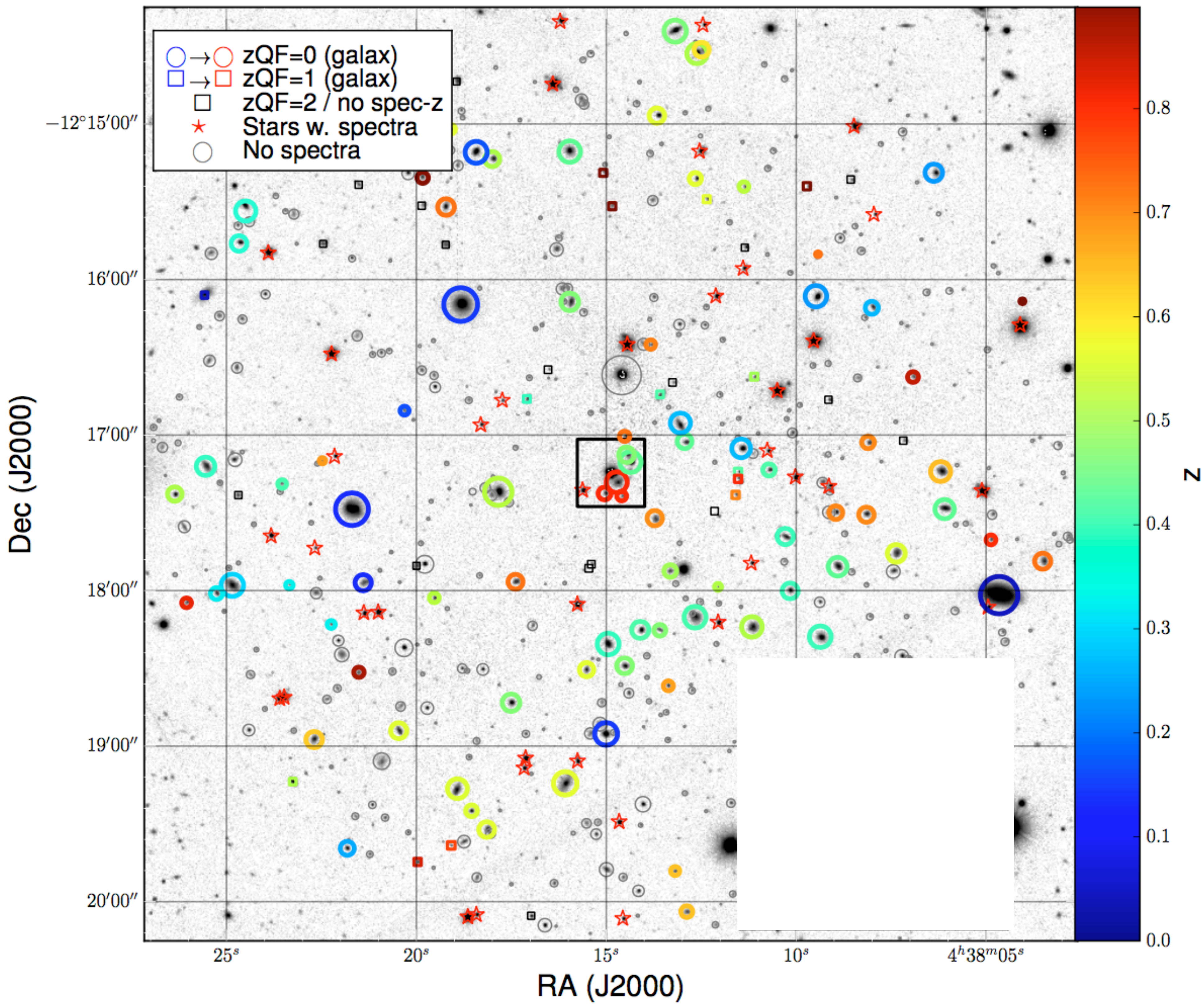
1. Gaussian Random Field/ Power spectrum/Correlation function/ Phase
2. BAO
3. Galaxy Clustering
4. RSD
5. Lensing: WL/ Strong Lensing
6. Linear Growth
7. Nonlinear growth (spherical collapse)
8. Halo model: Press-Schesther formalism, merge tree

Strong Lensing

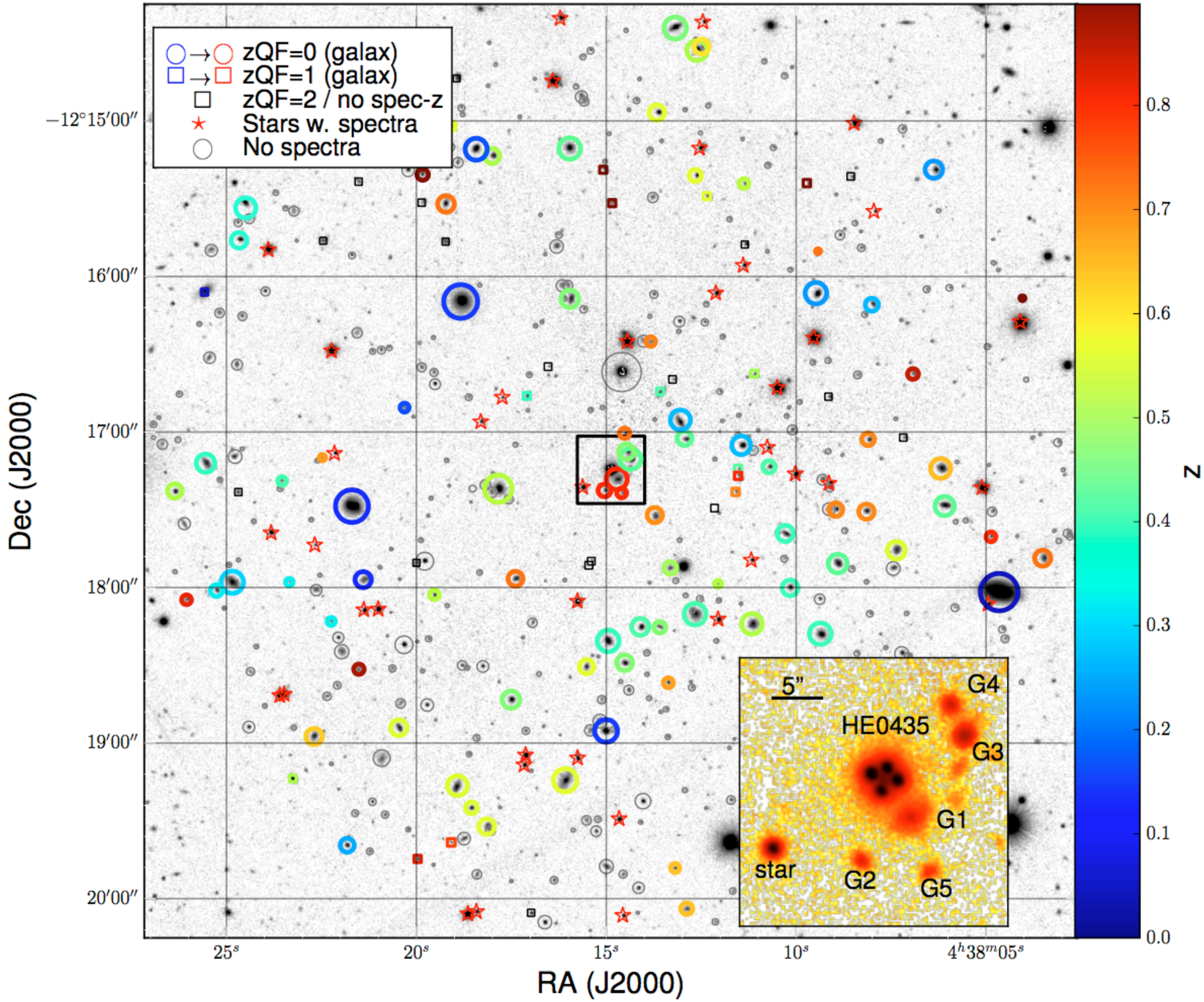
is a very tiny effect

3’x3’



zQF=0 (galax)
zQF=1 (galax)
zQF=2 / no spec-z
Stars w. spectra
No spectra
−12°15′00″
16′00″
17′00″
18′00″
19′00″
20′00″
Dec (J2000)
25s
20s
15s
10s
4h38m05s
RA (J2000)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
z

3’x3’



zQF=0 (galax)
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No spectra
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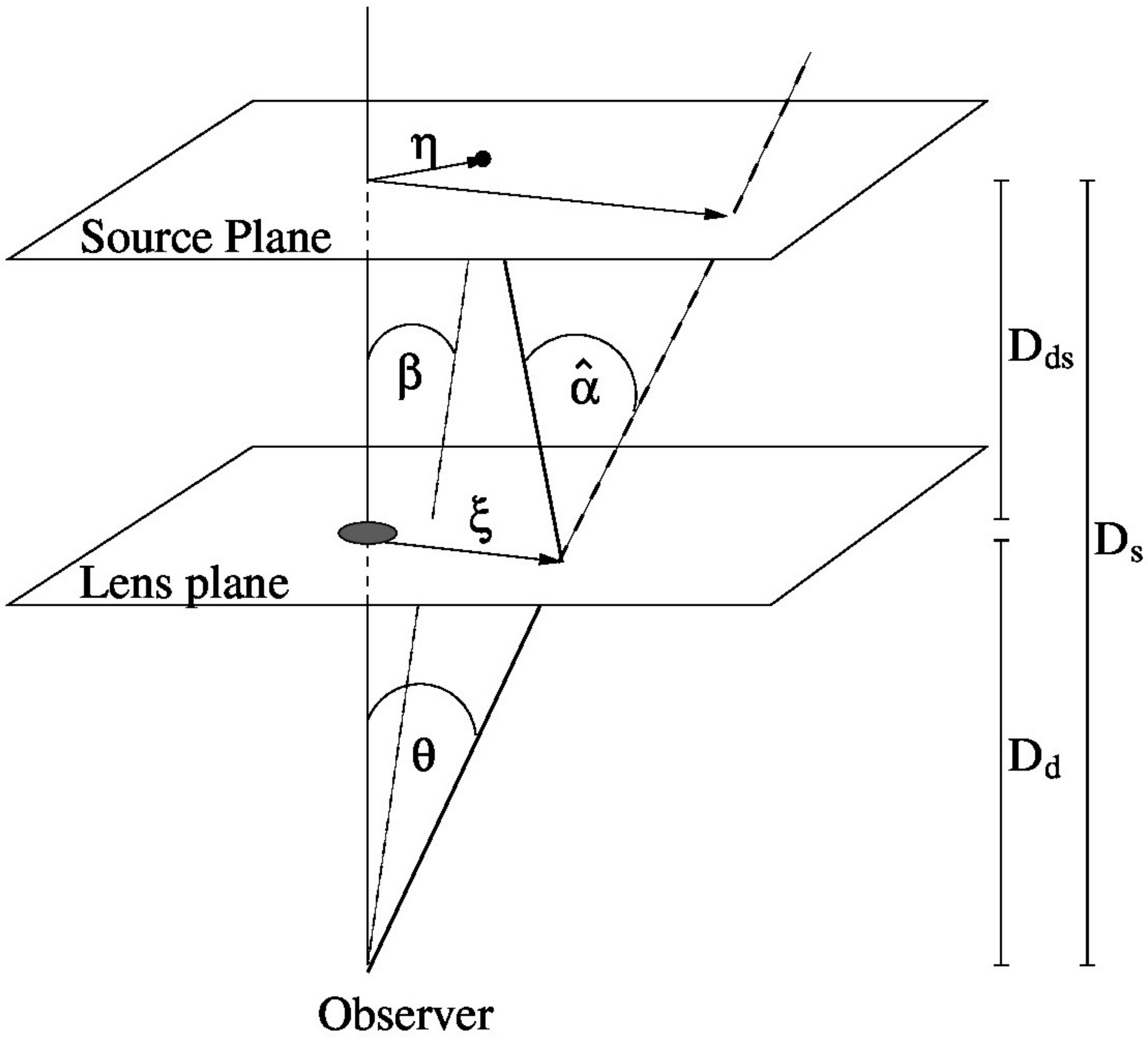
G2
G3
G4
G5
star

Lensing basics

[credit: Suyu]

Lens equation



[Schneider et al. 2006]

$$\boldsymbol{\eta} = \frac{D_\mathrm{s}}{D_\mathrm{d}}\boldsymbol{\xi} - D_\mathrm{ds}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi})$$

In terms of angular coord.:

$$\boldsymbol{\eta} = D_\mathrm{s}\boldsymbol{\beta}$$

$$\boldsymbol{\xi} = D_\mathrm{d}\boldsymbol{\theta}$$

$$\boxed{\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta})}$$

where

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{D_\mathrm{ds}}{D_\mathrm{s}}\hat{\boldsymbol{\alpha}}(D_\mathrm{d}\boldsymbol{\theta})$$

If $\vec{\theta}$, $\vec{\beta}$ and $\hat{\vec{\alpha}}$ are small, the true position of the source and its observed position on the sky are related by a very simple relation, obtained by a geometrical construction. This relation is called the *lens equation* and is written as

$$\vec{\theta} D_{\mathsf{S}} = \vec{\beta} D_{\mathsf{S}} + \hat{\vec{\alpha}} D_{\mathsf{LS}} \ , \tag{2.4}$$

where D_{LS} is the angular diameter distance between lens and source.
Defining the reduced deflection angle

$$\vec{\alpha}(\vec{\theta}) \equiv \frac{D_{\mathsf{LS}}}{D_{\mathsf{S}}} \hat{\vec{\alpha}}(\vec{\theta}) \ , \tag{2.5}$$

from Eq. (2.4), we obtain

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}) \ . \tag{2.6}$$

[credit: Suyu]

Deflection angle

Recall from General Relativity:

$$\hat{\boldsymbol{\alpha}} = \frac{4GM}{c^2 \xi}$$

For weak gravitational field and small deflection angles (*geometrically-thin lens*), a light ray with spatial trajectory ($\xi_1(\lambda)$, $\xi_2(\lambda)$, $r_3(\lambda)$) that passes through distribution with 3D density $\rho(\boldsymbol{r})$ will be deflected by

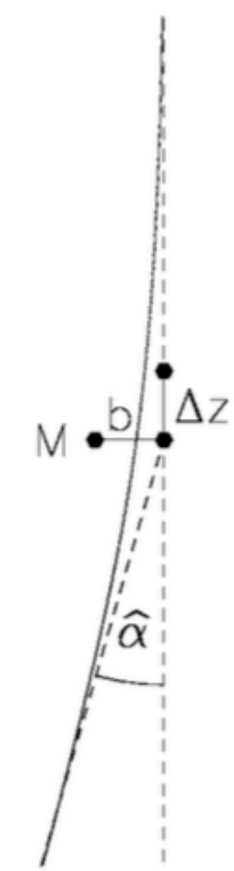
$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2} \int \mathrm{d}^2\xi' \underbrace{\int \mathrm{d}r_3' \, \rho(\xi_1', \xi_2', r_3')}_{\Sigma(\boldsymbol{\xi}')} \frac{\boldsymbol{\xi} - \boldsymbol{\xi}'}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2}$$

Born approx

$$\hat{\vec{\alpha}} = \frac{2}{c^2} \int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi \mathrm{d}\lambda \, . \qquad (1.35)$$

The deflection is thus the integral over the "pull" of the gravitational potential perpendicular to the light path. Note that $\vec{\nabla}\Phi$ points away from the lens centre, so $\hat{\vec{\alpha}}$ points towards it.

As it stands, the equation for $\hat{\vec{\alpha}}$ is not useful, as we would have to integrate over the actual light path. However, since $\Phi/c^2 \ll 1$, we expect the deflection angle to be small. Then, we can adopt the Born approximation familiar from scattering theory and integrate over the unperturbed light path.



Suppose, therefore, a light ray starts out into $+\vec{e}_z$-direction and passes a lens at $z = 0$, with impact parameter b. The deflection angle is then given by

$$\boxed{\hat{\vec{\alpha}}(b) = \frac{2}{c^2} \int_{-\infty}^{+\infty} \vec{\nabla}_\perp \phi \mathrm{d}z} \qquad (1.36)$$

[Pb: derive]

Special case: point mass lens

If the lens is a point mass, then

$$\Phi = -\frac{GM}{r} \tag{1.37}$$

with $r = \sqrt{x^2+y^2+z^2} = \sqrt{b^2+z^2}$, $b = \sqrt{x^2+y^2}$ and

$$\vec{\nabla}_\perp \phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \end{pmatrix} = \frac{GM}{r^3} \begin{pmatrix} x \\ y \end{pmatrix} . \tag{1.38}$$

The deflection angle is then

$$\begin{aligned} \hat{\vec{\alpha}}(b) &= \frac{2GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{(b^2+z^2)^{3/2}} \\ &= \frac{4GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \left[\frac{z}{b^2(b^2+z^2)^{1/2}} \right]_0^\infty = \frac{4GM}{c^2 b} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} , \end{aligned} \tag{1.39}$$

with

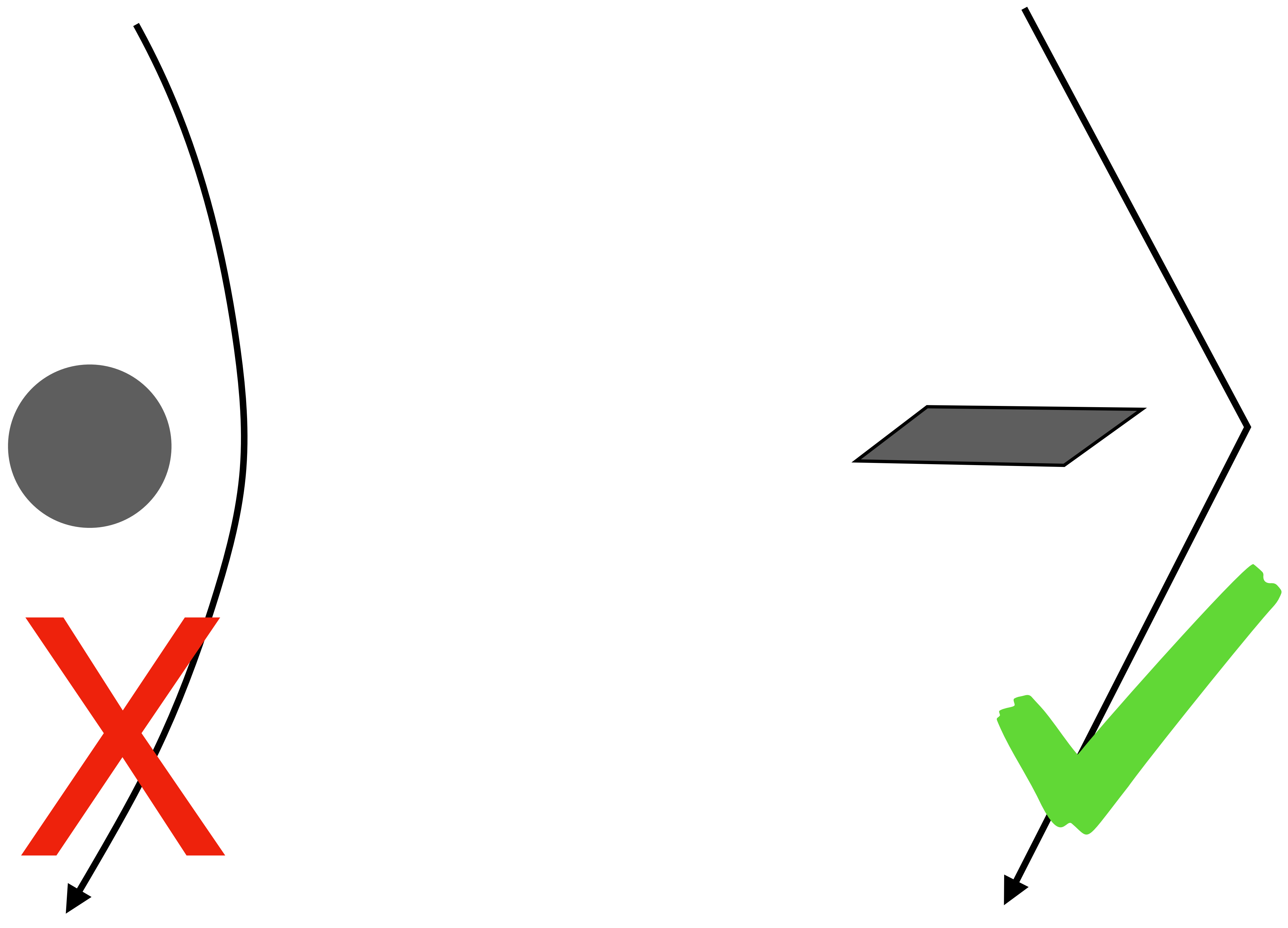
$$\begin{pmatrix} x \\ y \end{pmatrix} = b \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{1.40}$$

Notice that $R_s = \frac{2GM}{c^2}$ is the Schwarzschild radius of a (point) mass M, thus

$$|\hat{\vec{\alpha}}| = \frac{4GM}{c^2 b} = 2\frac{R_s}{b} . \tag{1.41}$$

Also notice that $\hat{\vec{\alpha}}$ is linear in M, thus the deflection angles of an array of lenses can linearly be superposed.
Note that the deflection angle found here in the framework of general relativity exceeds by a factor of two that calculated by using standard Newtonian Gravity (see Eq. 1.13), as anticipated at the beginning of this chapter.

Thin-lens approx



lens model

Point mass & SIS

3.1 Point masses

Let us begin with point masses as lenses. The deflection angle of a point mass was

$$\hat{\vec{\alpha}} = -\frac{4GM}{c^2 b}\vec{e}_r \,, \tag{3.1}$$

where $\vec{e}_r$ is the unit vector in radial direction. No direction is prefered in an axisymmetric situation like that, so we can identify $\vec{e}_r$ with one coordinate axis and thus reduce the problem to one dimension. Then

$$\hat{\alpha} = \frac{4GM}{c^2 b} = \frac{4GM}{c^2 D_\mathsf{L} \theta} \,, \tag{3.2}$$

where we have expressed the impact parameter by the angle θ, $b = D_\mathsf{L}\theta$.

The lensing potential is given by

$$\hat{\Psi} = \frac{4GM}{c^2}\frac{D_\mathsf{LS}}{D_\mathsf{L} D_\mathsf{S}} \ln|\vec{\theta}| \,, \tag{3.3}$$

as one can show using

$$\nabla \ln|\vec{x}| = \frac{\vec{x}}{|\vec{x}|^2} \,. \tag{3.4}$$

The lens equation reads

$$\beta = \theta - \frac{4GM}{c^2 D_\mathsf{L}\theta}\frac{D_\mathsf{LS}}{D_\mathsf{S}} \,. \tag{3.5}$$

With the definition of the Einstein radius,

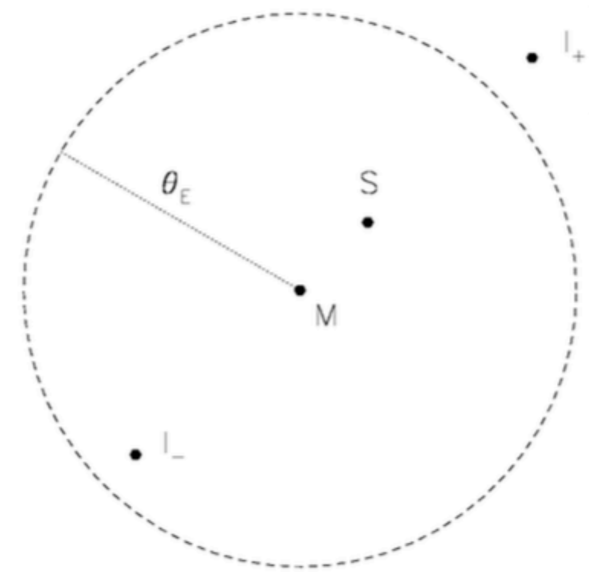
$$\theta_E \equiv \sqrt{\frac{4GM}{c^2}\frac{D_{\mathrm{LS}}}{D_{\mathrm{L}}D_{\mathrm{S}}}}\ , \tag{3.6}$$

we have

$$\beta = \theta - \frac{\theta_E^2}{\theta}\ . \tag{3.7}$$

Dividing by θ_E and setting $y = \beta/\theta_E$ and $x = \theta/\theta_E$, the lens equation in its adimensional form is written as

$$\boxed{y = x - \frac{1}{x}} \tag{3.8}$$



Multiplication with x leads to

$$x^2 - xy - 1 = 0\ , \tag{3.9}$$

which has two solutions:

$$x_\pm = \frac{1}{2}\left[y \pm \sqrt{y^2 - 4}\right]\ . \tag{3.10}$$

Thus, a point-mass lens has two images for any source, irrespective of its distance y from the lens. Why not three? Because its mass is singular and thus the time-delay surface is not continously deformed.

If $y = 0$, $x_\pm = \pm 1$; that is, a source directly behind the point lens has a ring-shaped image with radius θ_E. For order-of-magnitude estimates:

$$\begin{aligned}\theta_E &\approx (10^{-3})'' \left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D}{10\mathrm{kpc}}\right)^{-1/2}\ ,\\ &\approx 1'' \left(\frac{M}{10^{12}M_\odot}\right)^{1/2}\left(\frac{D}{\mathrm{Gpc}}\right)^{-1/2}\ ,\end{aligned} \tag{3.11}$$

Singular Isothermal Sphere

One of the two density profiles satisfying these sets of equations is given by

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2} \;, \tag{3.40}$$

where σ_v is the velocity dispersion of the "gas" particles and r is the distance from the sphere center. By projecting the three-dimensional density along the line of sight, we obtain the corresponding surface density

$$\begin{aligned}\Sigma(\xi) &= 2\frac{\sigma_v^2}{2\pi G}\int_0^\infty \frac{\mathrm{d}z}{\xi^2+z^2} \\ &= \frac{\sigma_v^2}{\pi G}\frac{1}{\xi}\left[\arctan\frac{z}{\xi}\right]_0^\infty \\ &= \frac{\sigma_v^2}{2G\xi} \;.\end{aligned} \tag{3.41}$$

This density profile has a singularity at $\xi = 0$, where the density is ideally infinite. Nevertheless, it has been used to describe the matter distribution in galaxies, expecially because it can reproduce the flat rotation curves of spiral galaxies.

[Pb: derive]

By choosing

$$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_\mathsf{L} D_\mathsf{LS}}{D_\mathsf{S}} \tag{3.42}$$

as the length scale on the lens plane, we obtain:

$$\Sigma(x) = \frac{\sigma_v^2}{2G\xi}\frac{\xi_0}{\xi_0} = \frac{1}{2x}\frac{c^2}{4\pi G}\frac{D_\mathsf{S}}{D_\mathsf{L} D_\mathsf{LS}} = \frac{1}{2x}\Sigma_\mathrm{cr} \, . \tag{3.43}$$

Thus, the convergence for the singular isothermal profile is

$$\kappa(x) = \frac{1}{2x} \, , \tag{3.44}$$

and the lensing potential (2.15) is

$$\Psi(x) = |x| \, . \tag{3.45}$$

Using Eqs. (2.16), we obtain

$$\alpha(x) = \frac{x}{|x|} \, , \tag{3.46}$$

and the lens equation reads

$$y = x - \frac{x}{|x|} \, . \tag{3.47}$$

If $y < 1$, two solutions of the lens equation exist. They arise at $x = y - 1$ and $x = y + 1$, on opposite sides of the lens center. The corresponding angular positions of the images are

$$\theta_\pm = \beta \pm \theta_E \tag{3.48}$$

where θ_E is the *Einstein radius*, defined now as

$$\theta_E = \sqrt{\frac{4GM(\theta_E)}{c^2}\frac{D_\mathsf{LS}}{D_\mathsf{L} D_\mathsf{S}}} \, . \tag{3.49}$$

The quantity $M(\theta_E)$ is the mass within the Einstein radius. The angular separation between the two images therefore is $\Delta(\theta) = 2\theta_E$: the Einstein radius defines a typical scale for separation between multiple images.

source model

Sersic

The Sersic profile

- Empirically devised by Sersic (1963) as a good fitting fn

$$\boxed{I(r) = I_0 \exp\left(-\left(\frac{r}{\alpha}\right)^{1/n}\right)}$$

$I(r)$ = intensity at radius r

I_0 = central intensity (intensity at centre)

α = scalelength (radius at which intensity drops by e^{-1})

n = Sersic index (shape parameter)

Can be used to describe most structures, e.g.,

Elliptical: $1.5 < n < 20$

Bulge: $1.5 < n < 10$

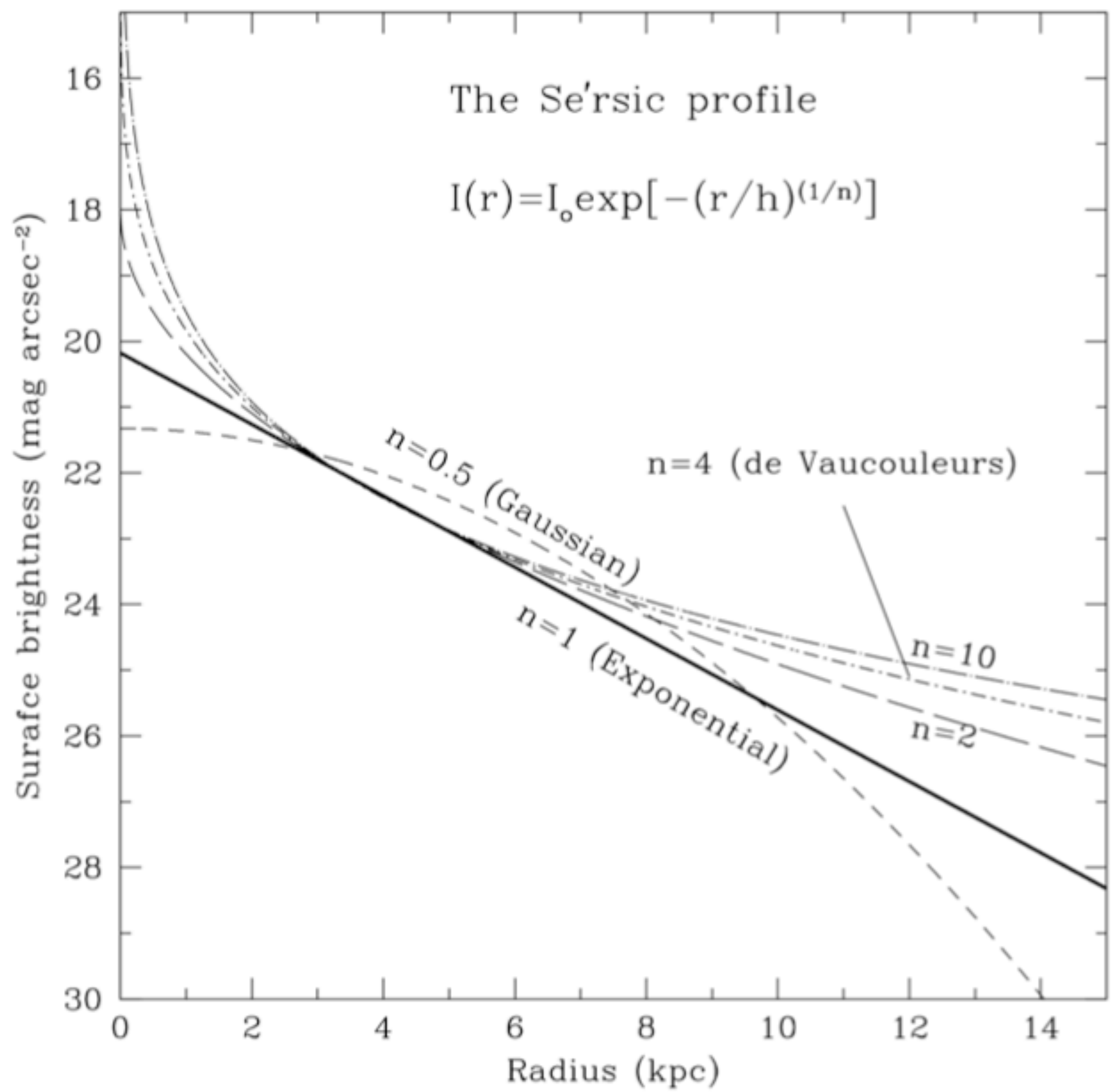
Pseudo-bulge: $1 < n < 2$

Bar: $n \sim 0.5$

Disc: $n \sim 1$

Total light profile = sum of components.

Sersic shapes



Caustics, Critical line

Jacobian matrix

[credit: Suyu]

Magnification

Lensing conserves surface brightness

Flux F = surface brightness x solid angle

Magnification = $F_{observed}$ / $F_{intrinsic}$ = $\boxed{d\Omega_{observed} / d\Omega_{intrinsic}}$

Define Jacobian matrix:

$$\mathcal{A}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\theta}} \qquad \text{with} \quad \mathcal{A}_{ij} = \frac{\partial \beta_i}{\partial \theta_j}$$

source plane

lens plane

Magnification factor is

$$\mu(\boldsymbol{\theta}) = \frac{1}{\det \mathcal{A}(\boldsymbol{\theta})}$$

- $\mu>0$: positive parity
- $\mu<0$: negative parity (mirror image of source)
- det A = 0 : critical points/curves

[credit: Suyu]

Image distortion I [deriv]

Rewrite Jacobian matrix:

$$\mathcal{A}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\theta}} = \left(\delta_{ij} - \frac{\partial^2 \psi(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}\right) = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

where γ_1 and γ_2 are the two components of shear

$$\gamma \equiv \gamma_1 + \mathrm{i}\gamma_2 = |\gamma|\mathrm{e}^{2\mathrm{i}\varphi}$$

$$\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22}) \qquad \gamma_2 = \psi_{,12}$$

Magnification in terms of κ and γ is:

$$\mu = \frac{1}{\det \mathcal{A}} = \frac{1}{(1-\kappa)^2 - |\gamma|^2}$$

[credit: Suyu]

Image distortion II

Surface brightness conservation:

$$\boxed{I(\boldsymbol{\theta}) = I^{(\mathrm{s})}[\boldsymbol{\beta}(\boldsymbol{\theta})]} \longleftarrow$$

Master eq. used for the observation!

To visualize distortion, consider locally linearized lens eq.:

$$\vec{\beta} = \mathcal{A}(\vec{\theta}_0)\vec{\theta}$$

linearised Lens eq.
derived from magnification eq.

const. matrix

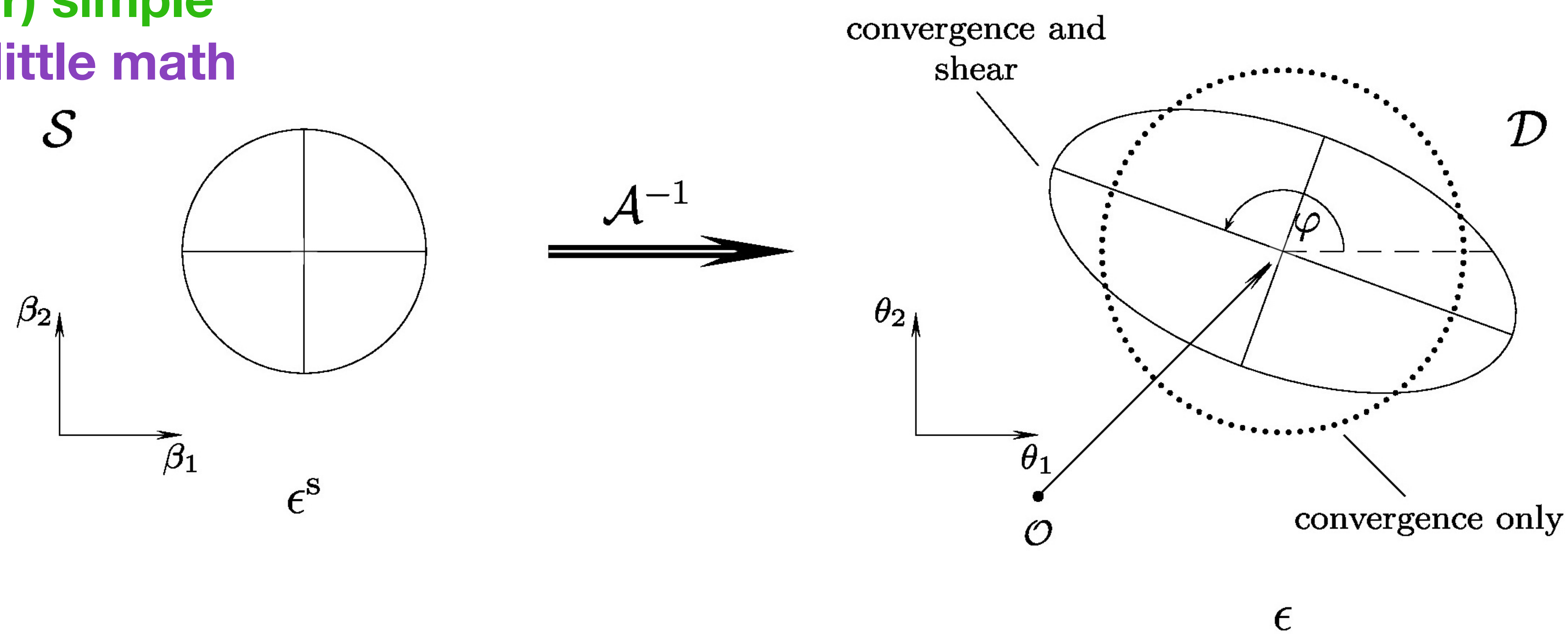
Question: for an infinitesimally small circular source, what would the shape of its lensed image be?

(1) Circular
(2) Elliptical
(3) Boxy
(4) Irregular
(5) None of the above

[credit: Suyu]

derive this picture!
(w/o shear) simple
(w. shear) little math

Image distortion III



Credit: M. Bradac

The lensed image of a small circular source with radius R is an ellipse

Major axis:

$$\frac{R}{1-\kappa-|\gamma|} = \frac{R}{(1-\kappa)(1-|g|)}$$

;

Minor axis:

$$\frac{R}{1-\kappa+|\gamma|} = \frac{R}{(1-\kappa)(1+|g|)}$$

reduced shear $g(\boldsymbol{\theta}) = \dfrac{\gamma(\boldsymbol{\theta})}{[1-\kappa(\boldsymbol{\theta})]}$

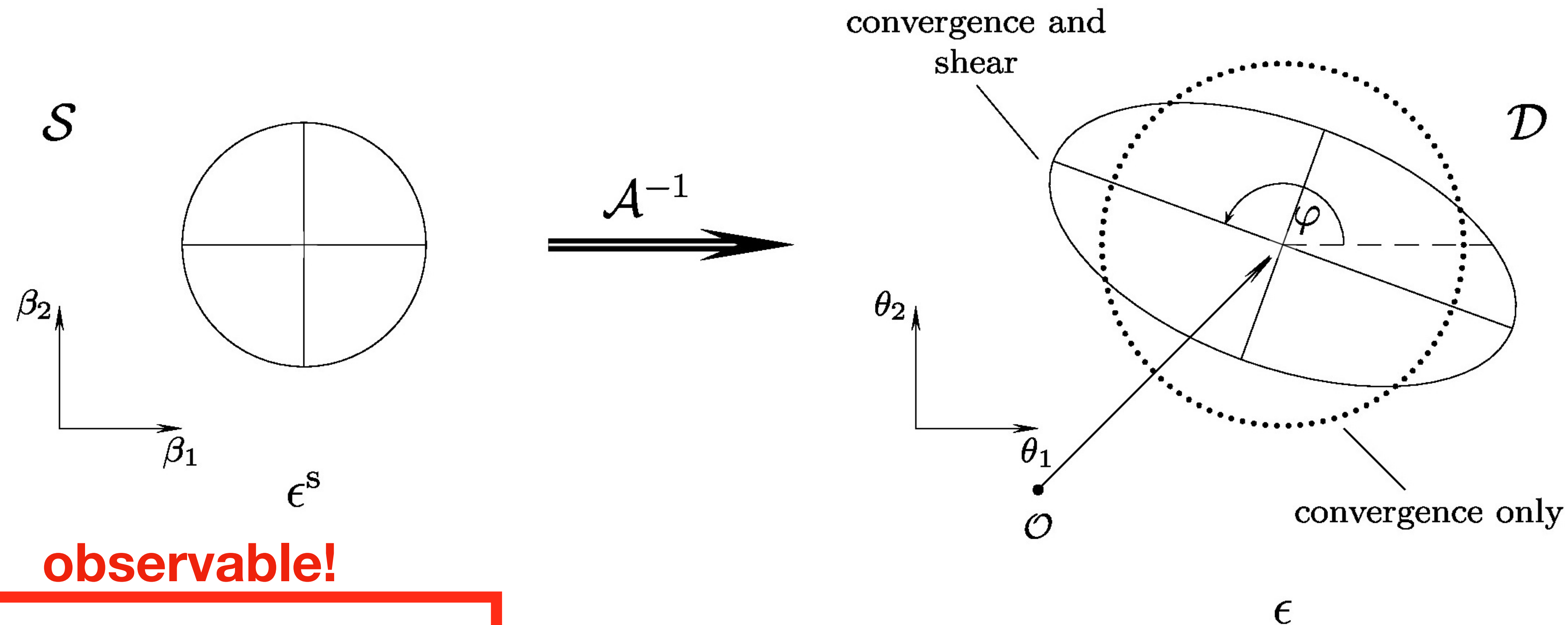
Angle of major axis from θ_1 the same as the shear angle φ

[Exercise: show these properties. Hint: try $\boldsymbol{\beta}(\lambda) = \cancel{\boldsymbol{\beta}_0} + R(\cos\lambda, \sin\lambda)$]

0

[credit: Suyu]

Image distortion IV



Credit: M. Bradac

observable!

Axis ratio of ellipse:

$$\frac{b}{a} = \frac{R}{(1-\kappa)(1+|g|)} \Big/ \frac{R}{(1-\kappa)(1-|g|)} = \frac{1-|g|}{1+|g|}$$

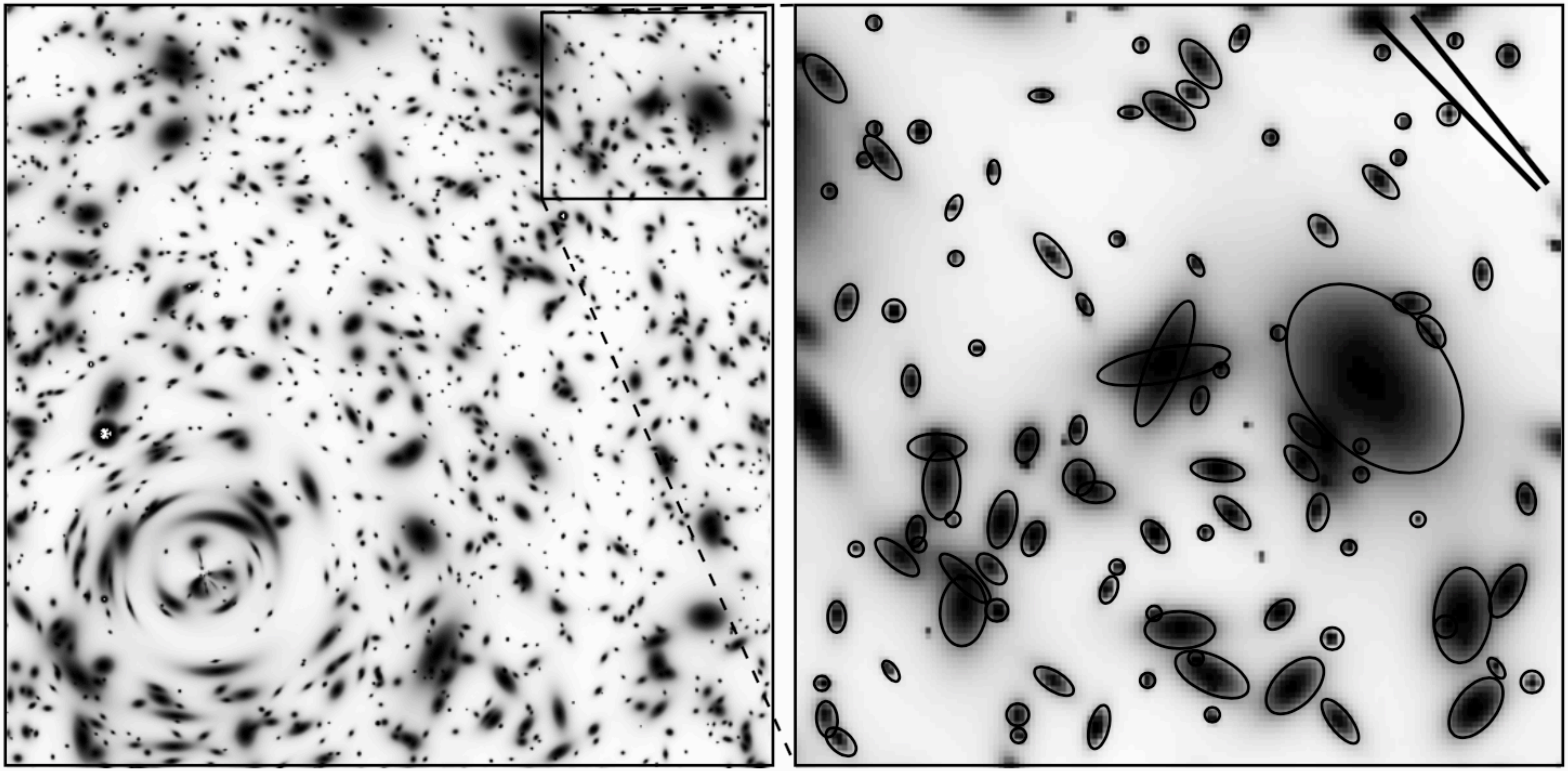
➡ shapes of lensed images yield estimate of reduced shear

$$|g| = \frac{1-b/a}{1+b/a}$$

BUT sources are not intrinsically round…

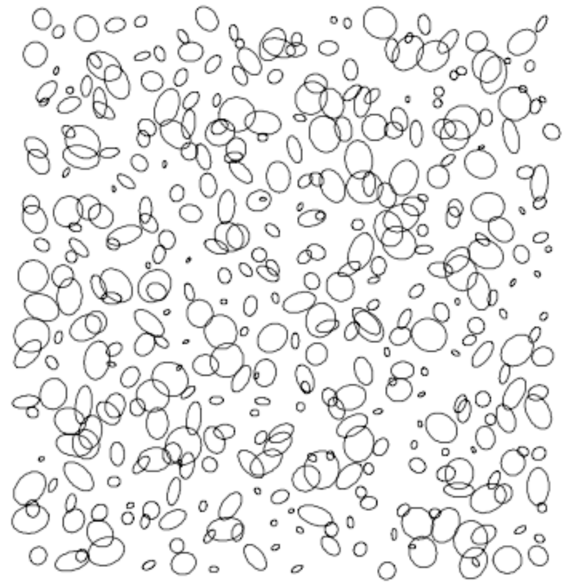
➡ average over many sources, and assume intrinsic ellipticities are *randomly oriented* angular size of lens is much larger than the angular size of the source!

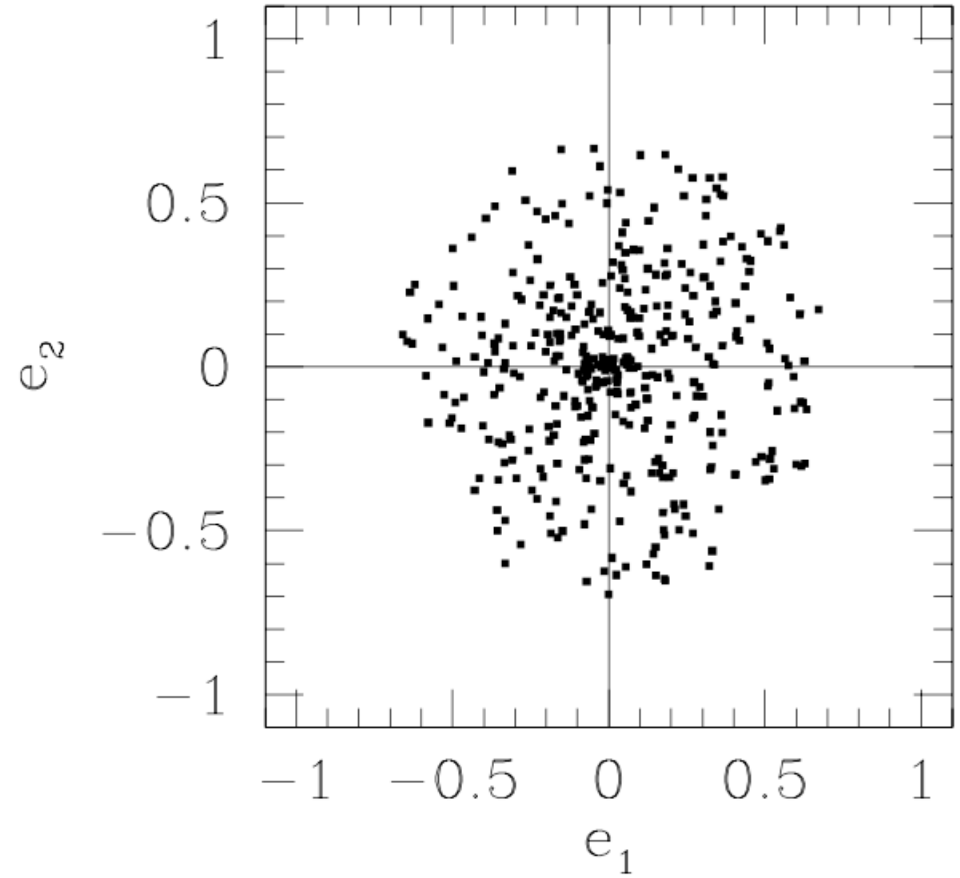
Ellipticity and local shear



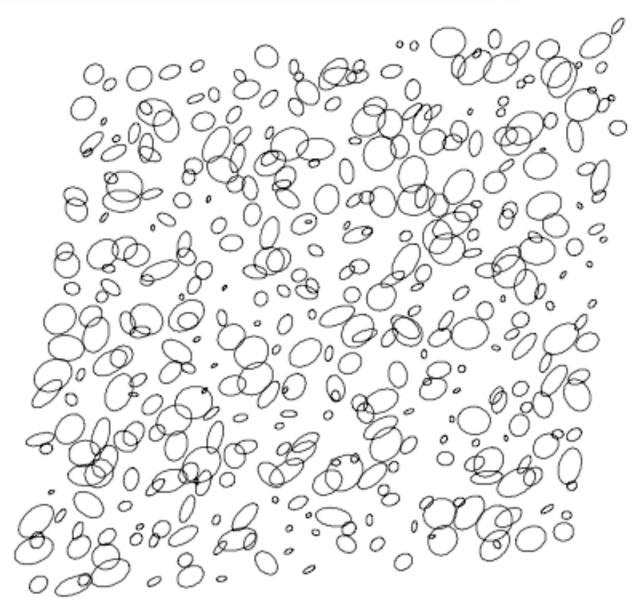
[from Y. Mellier]

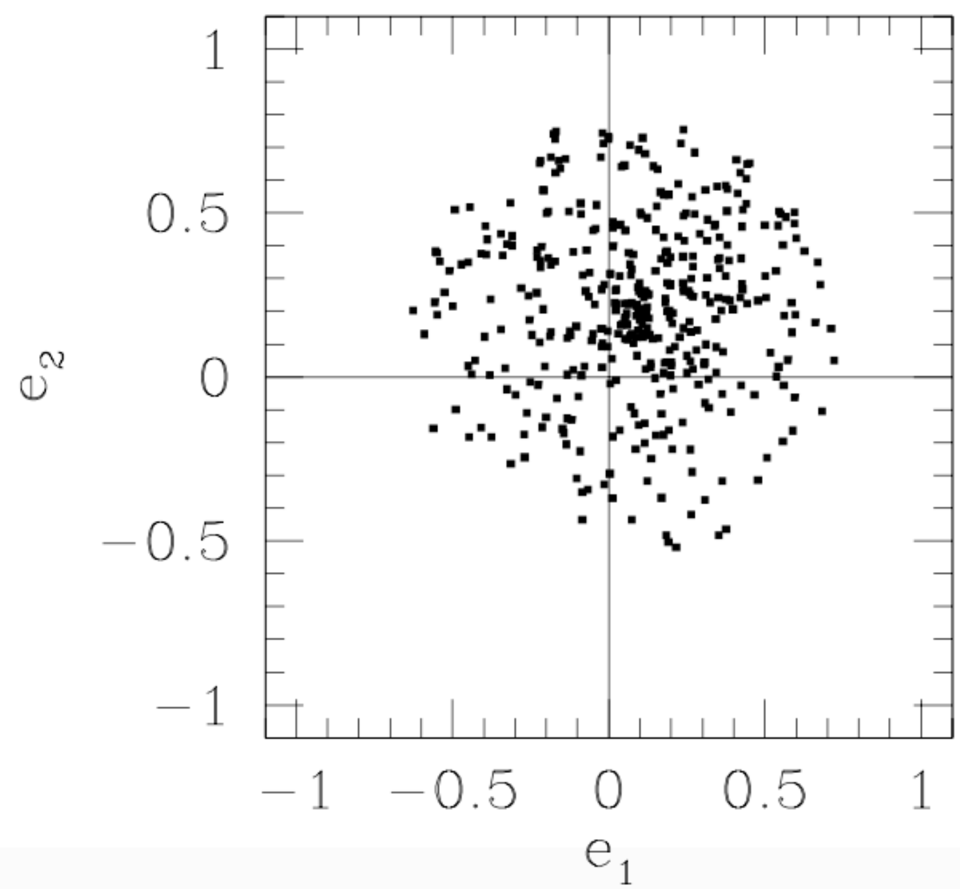
Galaxy ellipticities are an estimator of the local shear.





$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \frac{1}{I_{11} + I_{22}} \begin{pmatrix} I_{11} - I_{22} \\ 2I_{12} \end{pmatrix}$$



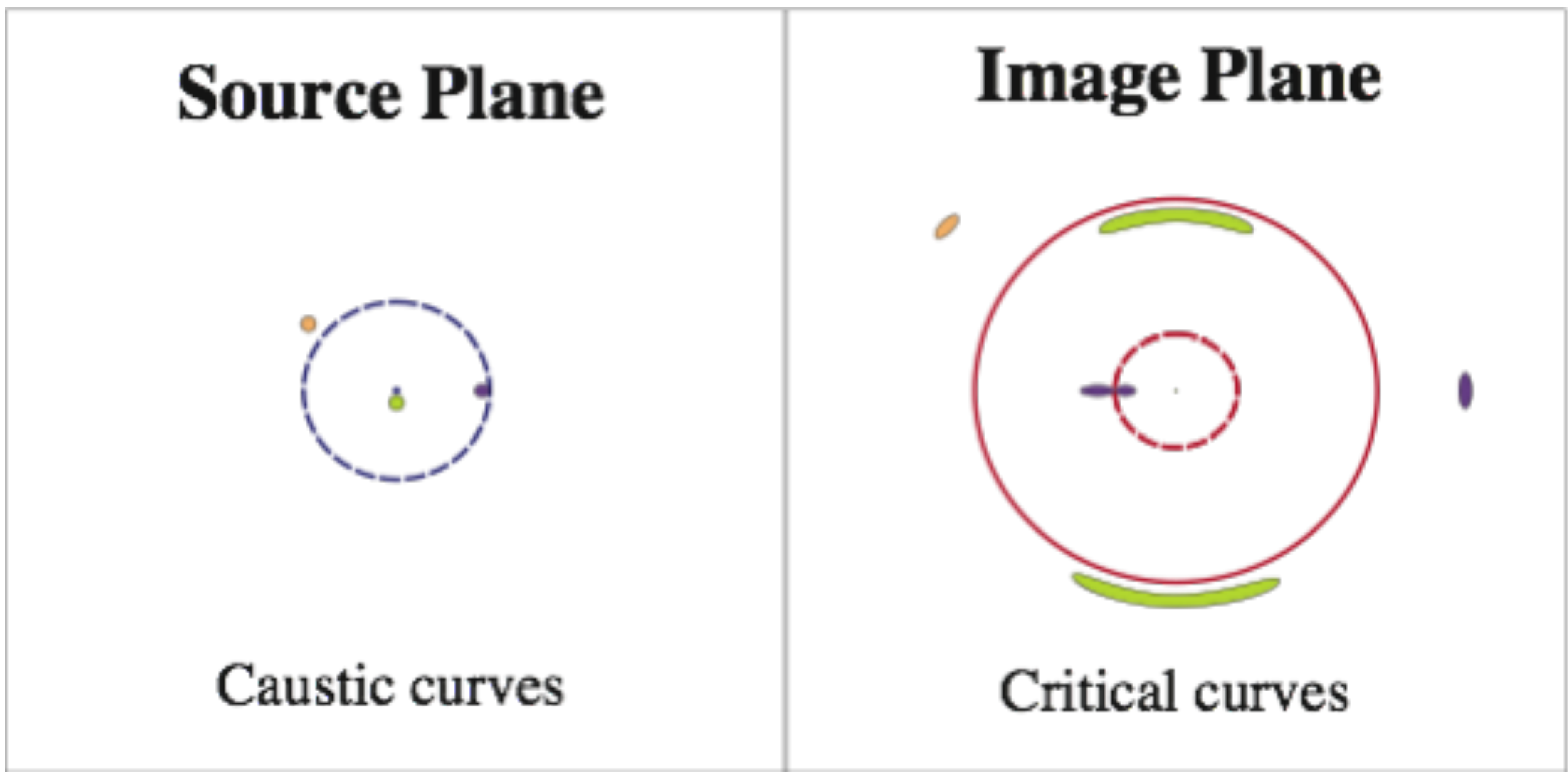


[credit: Suyu]

Critical curves and caustics I

det A = 0: critical curves on image plane θ
corresponds to caustics on source plane β

Example: non-singular isothermal sphere lens $\rho(r) = \dfrac{\rho_0}{r_c^2 + r^2}$



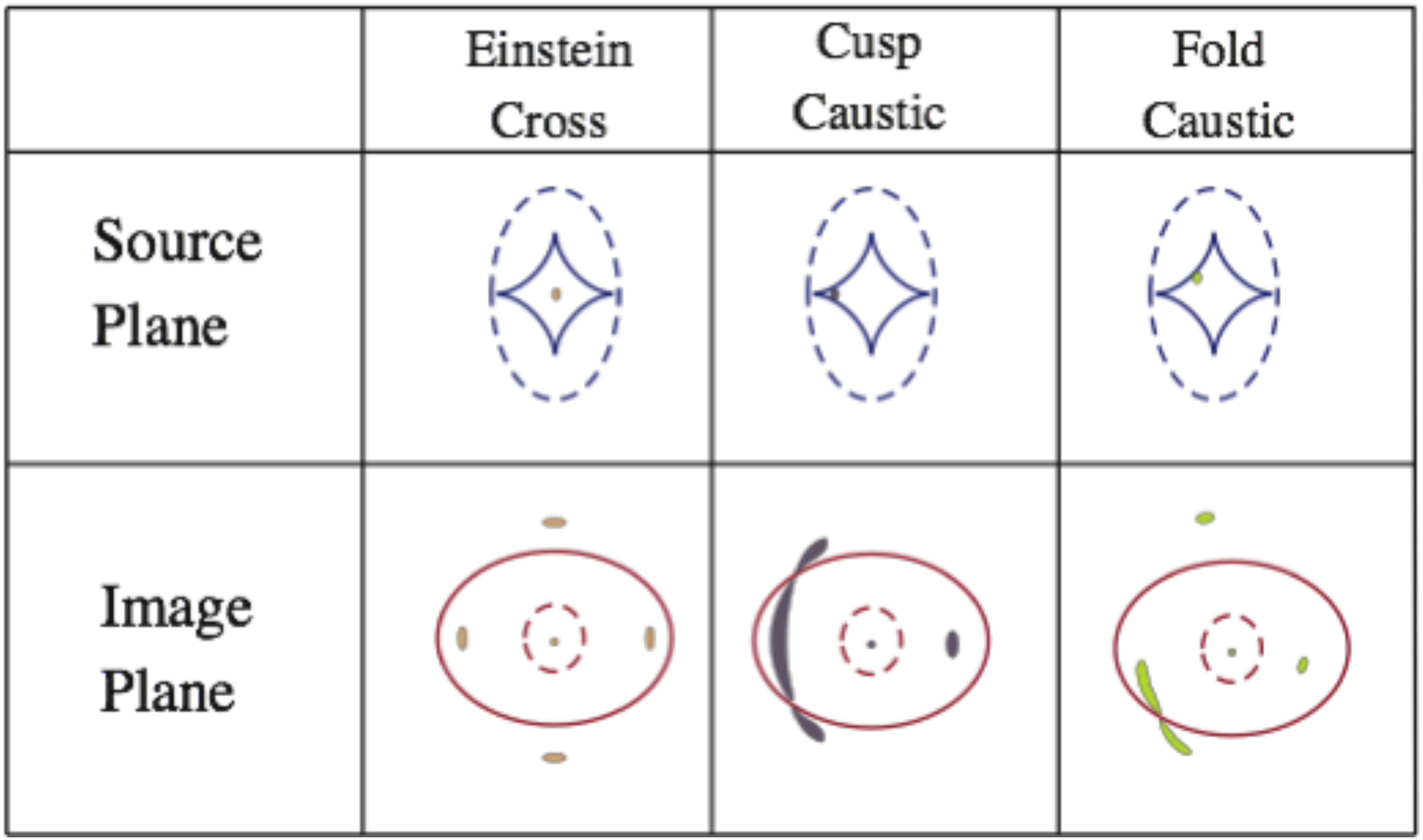
Credit: A. Amara & T. Kitching

[credit: Suyu]

suggest: visilens

Critical curves and caustics II

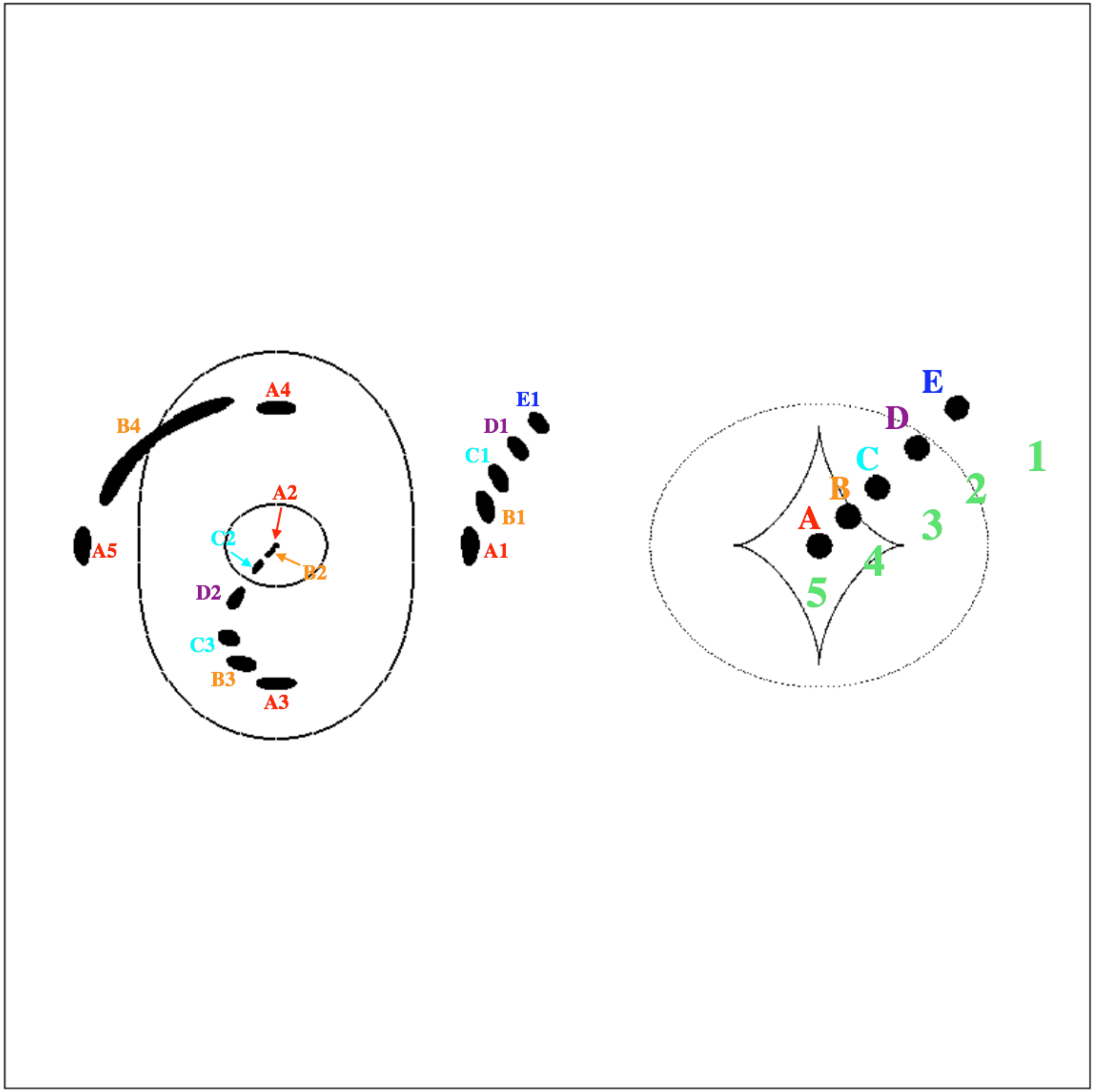
Example: non-singular isothermal ellipsoid lens



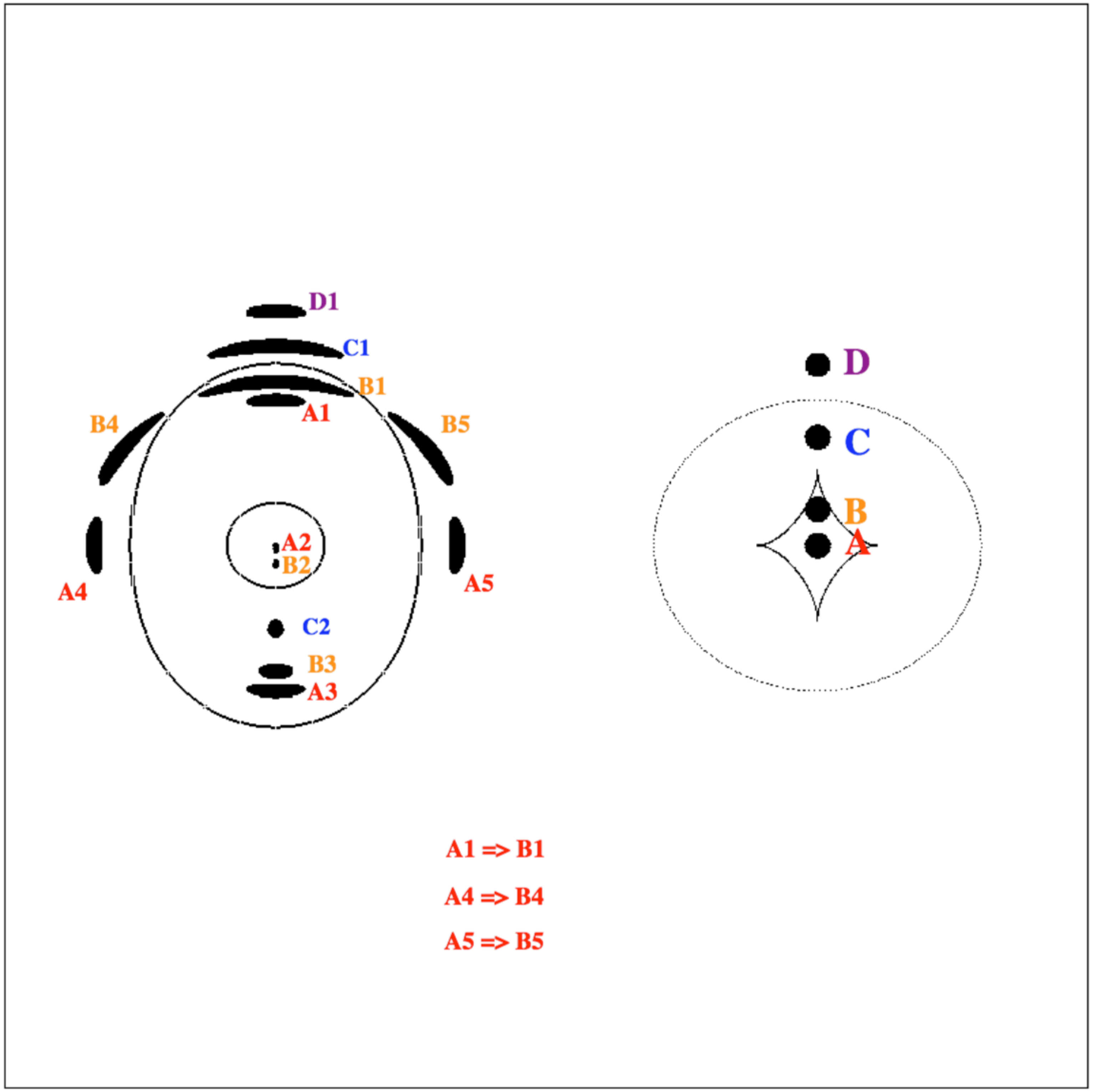
Credit: A. Amara & T. Kitching

demo from visilens

[https://arxiv.org/abs/astro-ph/9606001v2]



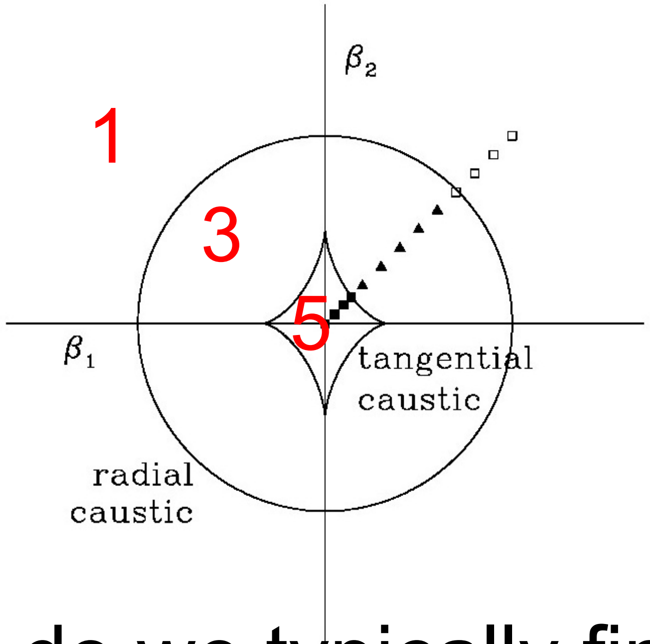
[https://arxiv.org/abs/astro-ph/9606001v2]

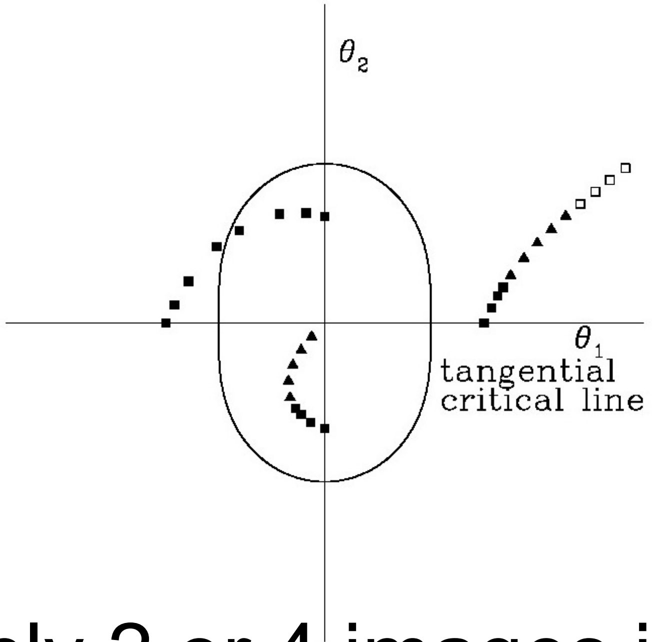


[credit: Suyu]

Critical curves and caustics VII

Caustics separate regions of different image multiplicity



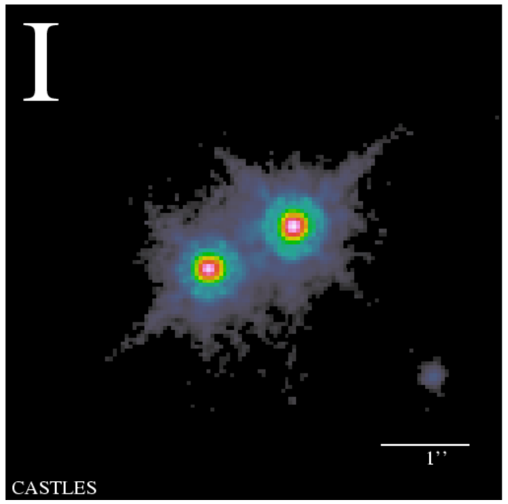


[Image credit:
C. Kochanek]

Why do we typically find only 2 or 4 images in real lens systems? *Ans.: central image is demagnified*

CASTLES

SBS0909+523



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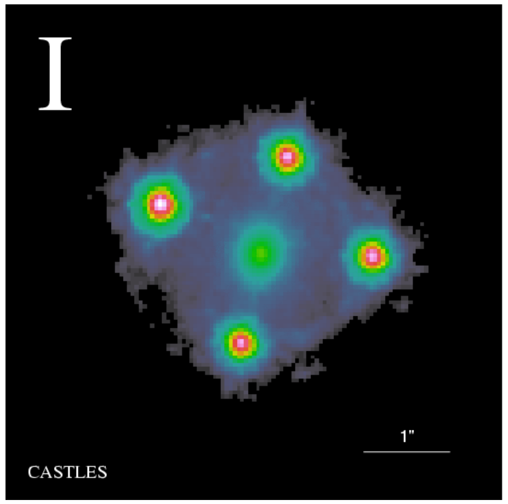

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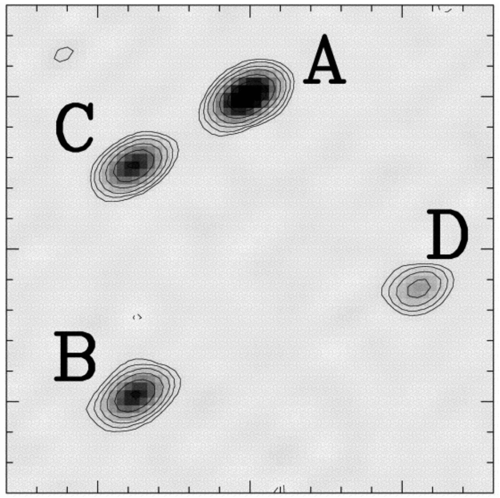


[Fassnacht et al. 2002]

Mass-Sheet degeneracy

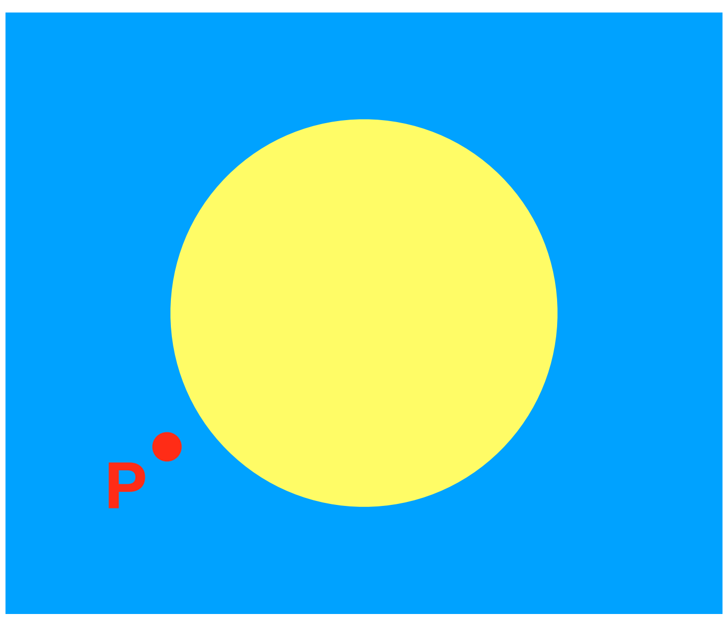
the *mass sheet degeneracy*: when adding to a given mass model, a sheet of constant mass density (i.e., constant convergence , as defined in the preceding chapters), one does not change any of the observables, except for the time-delay.

constant density ==>
constant convergence
!= constant potential

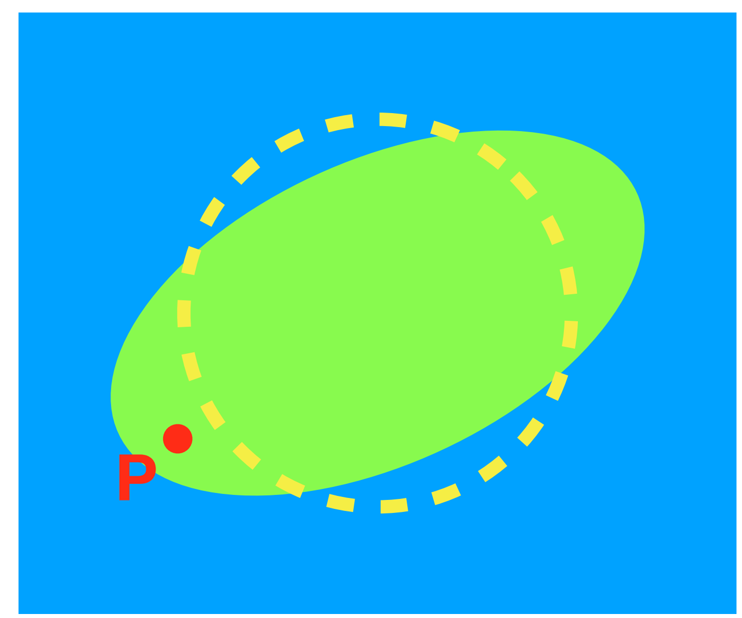
$$\nabla^2\Phi = 4\pi G\rho$$

$$\kappa = 4\pi G\rho$$

The additional mass can be internal to the lensing galaxy (e.g., ellipticity does not change the total mass within the Einstein radius, but does change κ at the position of the images) or due to intervening objects along the line of sight.



$$\kappa_p \propto \rho_p$$



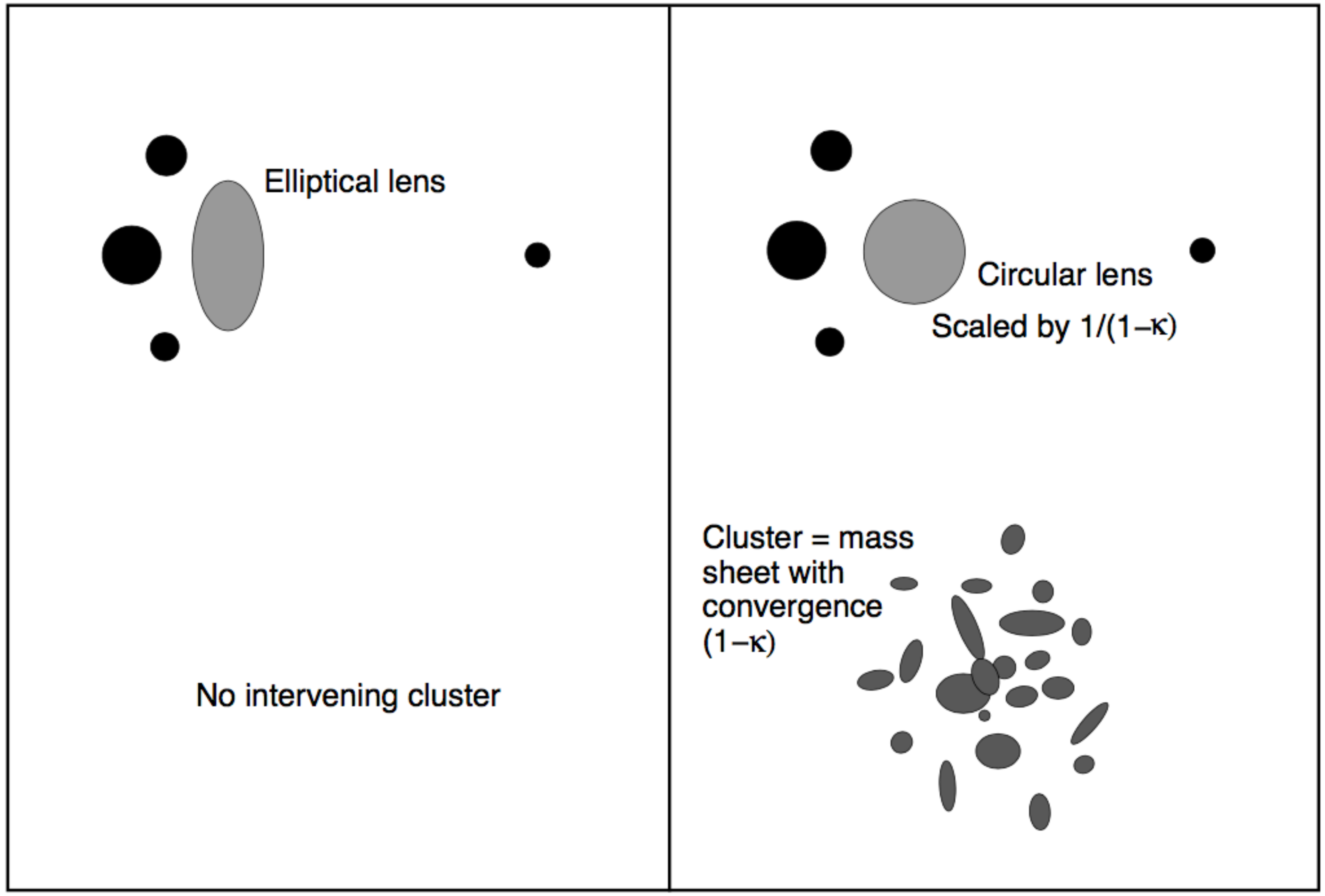


Figure 4. Two ways of obtaining a given image configuration. The left panel displays a system with four images, with an elliptical lens that introduces convergence and shear at the position of the images. On the right panel, is shown the same image geometry and flux ratios, but the lens is now circular. One would in principle only obtain two images with such a lens. The shear required to obtain four images is introduced by the nearby cluster. The mass density of the cluster is represented through its convergence κ. The mass of the main lens is scaled accordingly by $1/(1\text{-}\kappa)$ so that the image configuration remains the same as in the left panel: the mass in the main lens and in the cluster are degenerate. If no independent measurement is available for at least one of the components (main lens or cluster), it is often difficult to know, from the modeling alone, what exactly are their respective contributions.

The exact mass introduced by the mass sheet **increases the total mass of the lens**, but one can **re-scale** it and **locally change its slope** at the position of the images. The result is that the **image configuration does not change**, but **the convergence κ, at the position of the images does change**, and modifies the time-delay.

Therefore, knowledge of the the slope of the mass profile of the lensing galaxy, whether it be under the form of a model or of a measurement, is one of the keys to the determination of a “good” model.

Changing the slope of the lens will change κ at the position of the images, but adding intervening objects along the line of sight to the lens has a similar effect. A group or cluster of galaxies, located angularly close to the lens, will add its own contribution to the total mass density at the position of the images. If the group/cluster has a constant density κ, rescaling the total mass of the lensing galaxy by $1/(1 - \kappa)$ will leave the observed images configuration unchanged.

degeneracies can be broken or, at least, their effect can be strongly minimized, by constraining in an independent way (1) the mass profile of the main lens, and (2) the total mass (and possibly also the radial mass profile) of any intervening cluster along the line of sight. This work can be done with detailed imaging, spectroscopy of all objects along the line of sight, and by using numerical multi-components models for the total lensing potential.

[Credit: Schneider]

The unfortunate mass-sheet degeneracy

For a given source and lens redshift:

The mass distributions $\kappa(\boldsymbol{\theta})$ and, for all λ,

$$\kappa_\lambda(\boldsymbol{\theta}) := \lambda\kappa(\boldsymbol{\theta}) + (1-\lambda)$$

yield **the same** image configurations, magnification ratios, image shapes!

Magnification depends on λ, $\mu_\lambda = \mu/\lambda^2$ – but unmeasurable without information about the source (or source population)

[Time-delay affected, $(H_0\,\Delta t)_\lambda = \lambda(H_0\,\Delta t)$]

Radial slope of density profile affected

Invariant: (Mass inside) Einstein radius, angular structure (e.g., ellipticity)

Thus:
To determine slope of mass profile, absolute masses (away from the Einstein radius), Hubble constant, mass-sheet degeneracy must first be broken!!

$$\vec{\beta} + \vec{\alpha} = \vec{\theta}$$

$$\vec{\beta}_\lambda + \vec{\alpha}_\lambda = \vec{\theta}$$

[credit: Suyu]

Mass-sheet degeneracy I

Given a lens mass distribution $\kappa(\theta)$ with potential $\psi(\theta)$
Consider the following transformation:

$$\psi_\lambda(\boldsymbol{\theta}) = \frac{\lambda}{2}|\boldsymbol{\theta}|^2 + \underbrace{\boldsymbol{s} \cdot \boldsymbol{\theta}}_{\text{corresponding to constant shift on source plane (unobservable)}} + c + (1 - \lambda)\psi(\boldsymbol{\theta})$$

zero point of lens potential (unobservable)

Transformed deflection angle ($=\nabla\psi_\lambda$):

$$\boldsymbol{\alpha}_\lambda(\boldsymbol{\theta}) = \lambda\boldsymbol{\theta} + \boldsymbol{s} + (1 - \lambda)\boldsymbol{\alpha}(\boldsymbol{\theta})$$

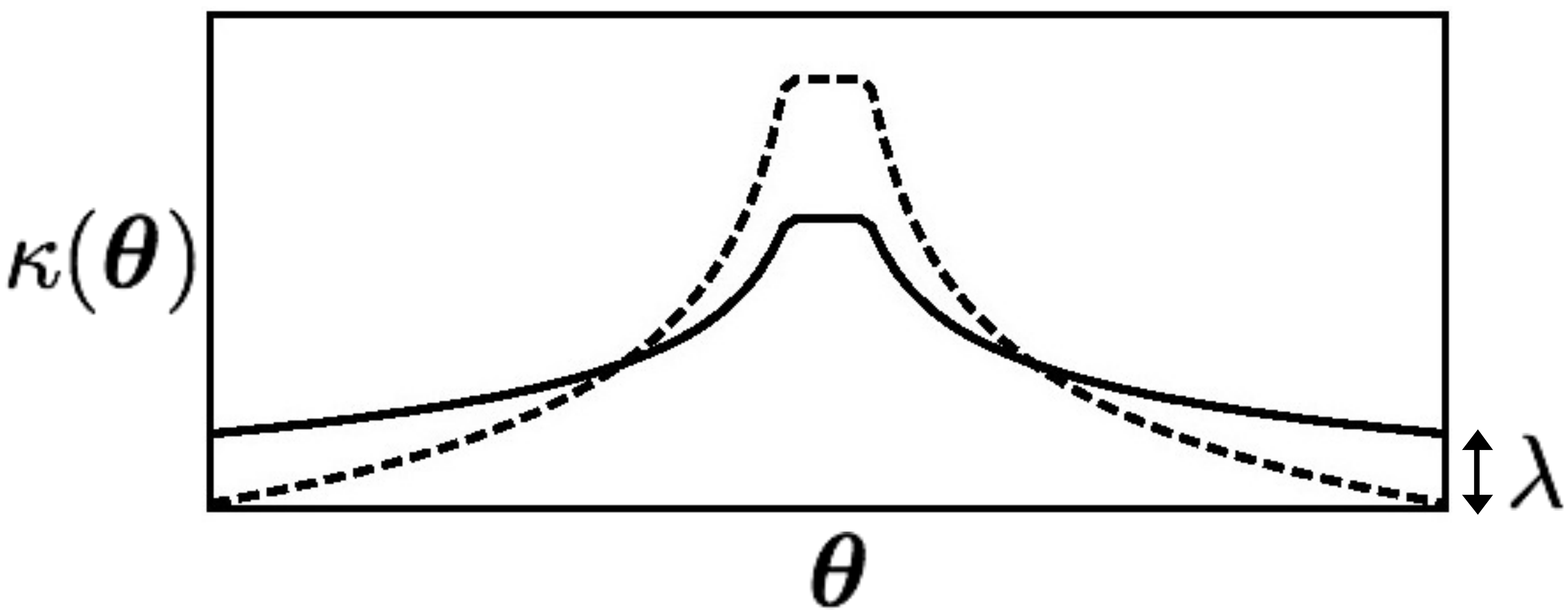
Transformed convergence ($=\nabla^2\psi_\lambda/2$):

$$\kappa_\lambda(\boldsymbol{\theta}) = \lambda + (1 - \lambda)\kappa(\boldsymbol{\theta})$$

[credit: Suyu]

Mass-sheet degeneracy II

Last slide: $\kappa_\lambda(\boldsymbol{\theta}) = \lambda + (1-\lambda)\kappa(\boldsymbol{\theta})$



Lens equation:

$$\boldsymbol{\beta}_\lambda = \boldsymbol{\theta} - \boldsymbol{\alpha}_\lambda(\boldsymbol{\theta}) = \boldsymbol{\theta} - \lambda\boldsymbol{\theta} - \boldsymbol{s} - (1-\lambda)\boldsymbol{\alpha}(\boldsymbol{\theta})$$

$$\Rightarrow \underbrace{\frac{\boldsymbol{\beta}_\lambda}{1-\lambda} + \frac{\boldsymbol{s}}{1-\lambda}} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta}) \equiv \boldsymbol{\beta}$$

source scaled and shifted, both unobservable $\Rightarrow$ degeneracy

[credit: Suyu]

Mass-sheet degeneracy III

$$\frac{\boldsymbol{\beta}_\lambda}{1-\lambda} + \frac{\boldsymbol{s}}{1-\lambda} = \boldsymbol{\beta}$$

source scaled and shifted, both effects unobservable

Magnification

$$\mathcal{A}_\lambda = (1-\lambda)\mathcal{A} \quad \Rightarrow \quad \mu_\lambda = \frac{\mu}{(1-\lambda)^2}$$

Recall

$$\mathcal{A} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} \gamma_\lambda(\boldsymbol{\theta}) &= (1-\lambda)\gamma(\boldsymbol{\theta}) \\ (1-\kappa_\lambda) &= (1-\lambda)(1-\kappa) \end{aligned}\right\}$$

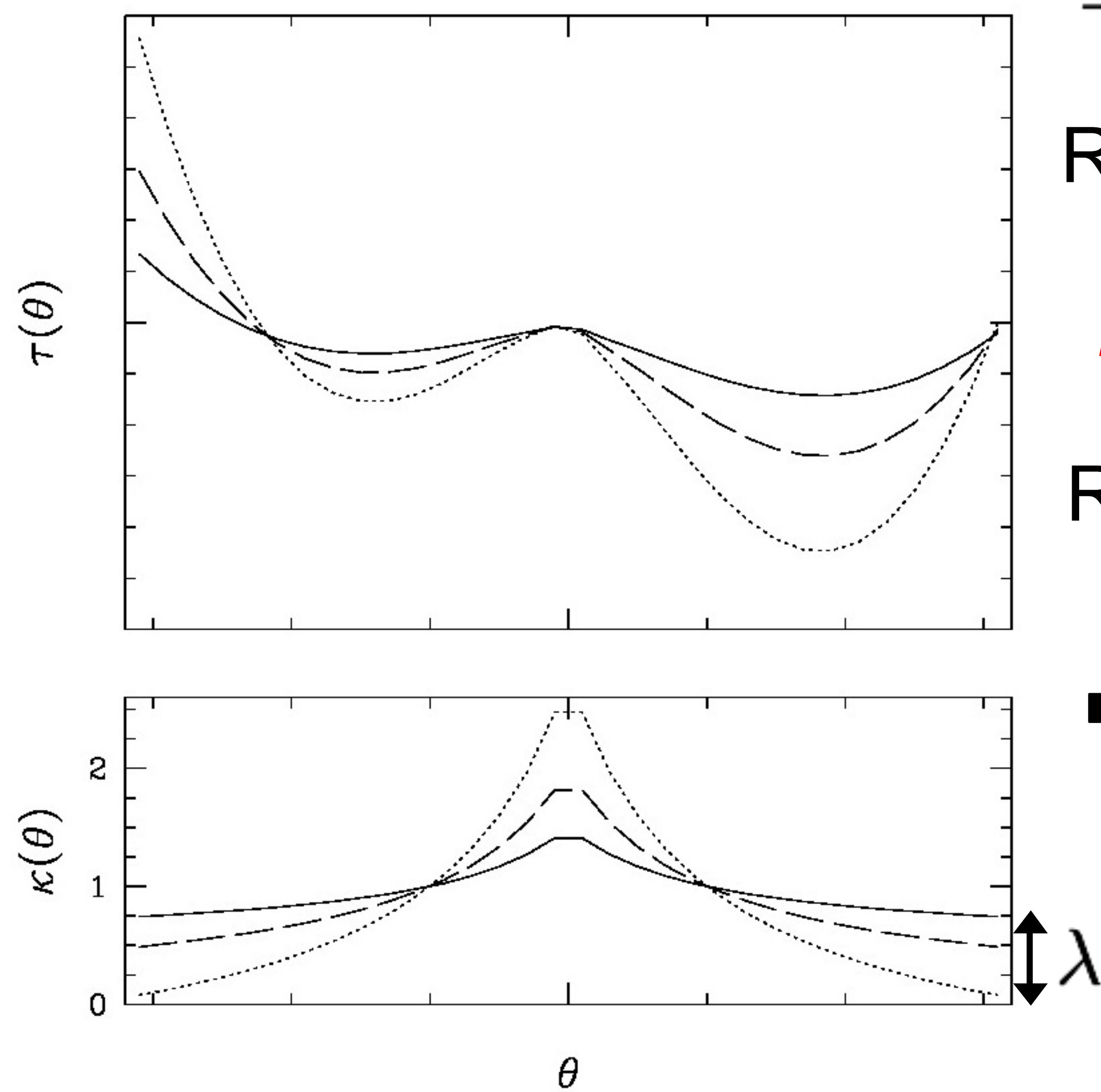
Reduced shear invariant

$$g_\lambda = \frac{\gamma}{1-\kappa} = g$$

Mass-sheet degeneracy IV

Fermat potential: $\tau_\lambda(\boldsymbol{\theta};\boldsymbol{\beta}) = \frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\beta}_\lambda)^2 - \psi_\lambda(\boldsymbol{\theta})$

$$= (1-\lambda)\tau(\boldsymbol{\theta};\boldsymbol{\beta}) + \mathrm{constant}$$



[Courbin et al. 2002]

Recall $\kappa_\lambda(\boldsymbol{\theta}) = \lambda + (1-\lambda)\kappa(\boldsymbol{\theta})$

Big impact on cosmography!

Recall $\Delta t(\boldsymbol{\theta};\boldsymbol{\beta}) = \frac{D_{\Delta t}}{c}\Delta\tau(\boldsymbol{\theta};\boldsymbol{\beta})$

➡ For fixed Δt,

model

$$D_{\Delta t,\,\lambda} = \frac{D_{\Delta t}}{1-\lambda}$$

True (including external convergence)

Further Reading

[https://arxiv.org/abs/astro-ph/9606001v2]

- **Introduction to Gravitational Lensing**

Lecture scripts by Massimo Meneghetti

Lensing Basics: II/III. Basic Theory by Sherry Suyu (slide)

Weak Gravitational Lensing & Cosmic Shear by Peter Schneider (slide)