Cosmic Large-scale Structure Formations

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18 weeks

outline

Background (1 w)

- universe geometry and matter components (1 hr)
- Standard candle (SNIa) (0.5 hr)
- Standard ruler (BAO) (0.5 hr)

Linear perturbation (9 w)

- relativistic treatment perturbation (2 hr)
- primordial power spectrum (2 hr)
- linear growth rate (2 hr)
- galaxy 2-pt correlation function (2 hr)
- Baryon Acoustic Oscillation (BAO) (2 hr)
- Redshift Space Distortion (RSD) (2 hr)
- Weak Lensing (2 hr)
- Einstein-Boltzmann codes (2 hr)

Non-linear perturbation (6 w)

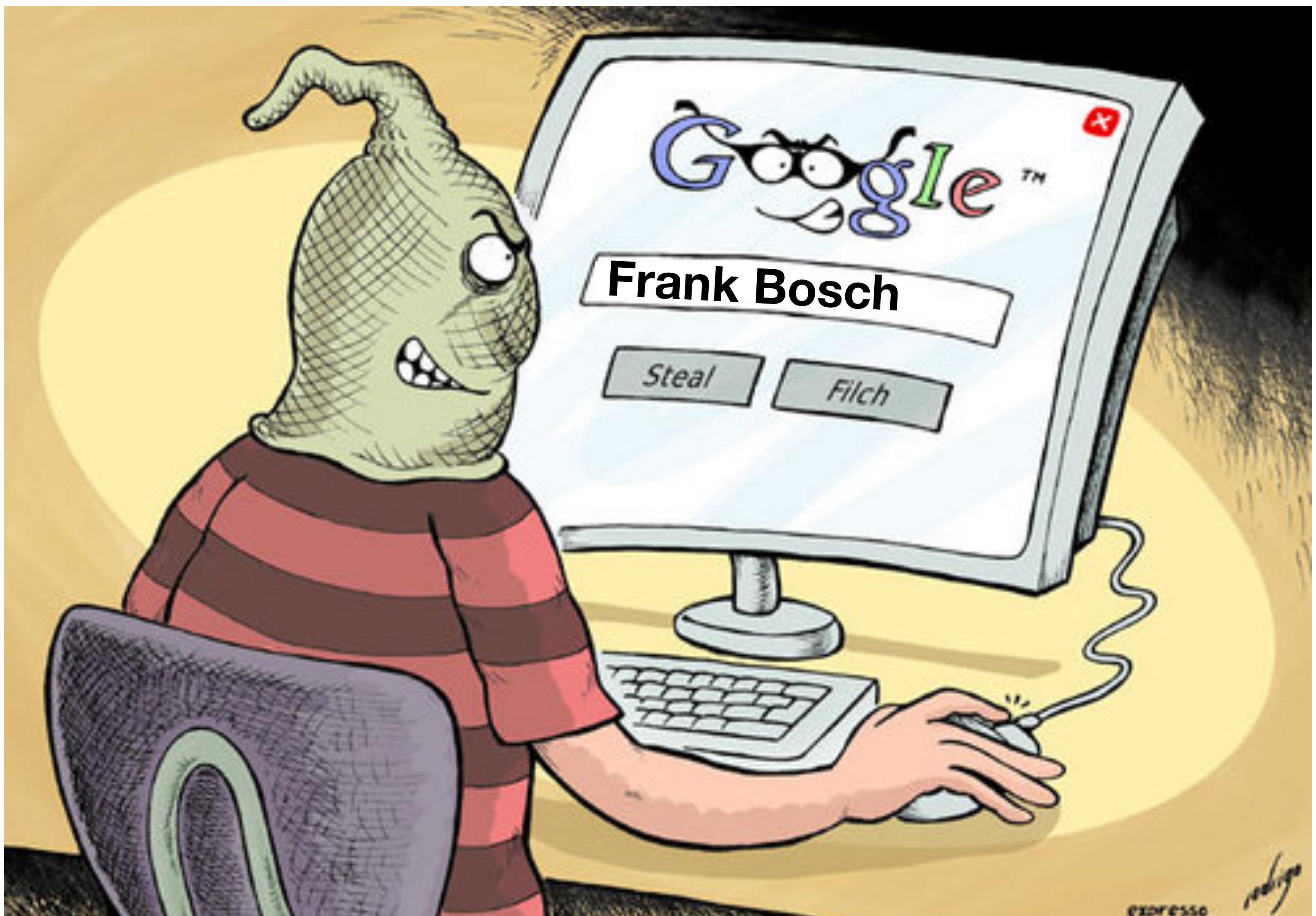
- Non-linear power spectrum (2 hr)
- halo model (2 hr)
- N-body simulation algorithms (2 hr)
- Press-Schechter (PS) halo mass function (2 hr)
- Extended-PS (EPS) halo mass function (2 hr)
- halo bias & halo density profile (2 hr)

Statistical analysis (2 w)

- Monte-Carlo Markov Chain sampler (2 hr)
- CosmoMC use (2 hr)

In this lecture, we will demonstrate the non-linear structure formation via the Spherical Collapse of Dark Matter particles.

In the following slides, I will heavily steal the lecture from F. Bosch

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1. Single Halo Collapsing Dynamics

Non-Linear Evolution

In the linear regime ($\delta \ll 1$) we can calculate the evolution of a density field of arbitrary form using linear perturbation theory.

In the non-linear regime ($\delta > 1$) perturbation theory is no longer valid. Modes start to couple to each other, and one can no longer describe the evolution of the density field with a simple growth rate: in general, no analytic solutions exist...

Because of this mode-coupling, the density field looses its Gaussian properties, i.e., in the non-linear regime, we no longer have a Gaussian random field.

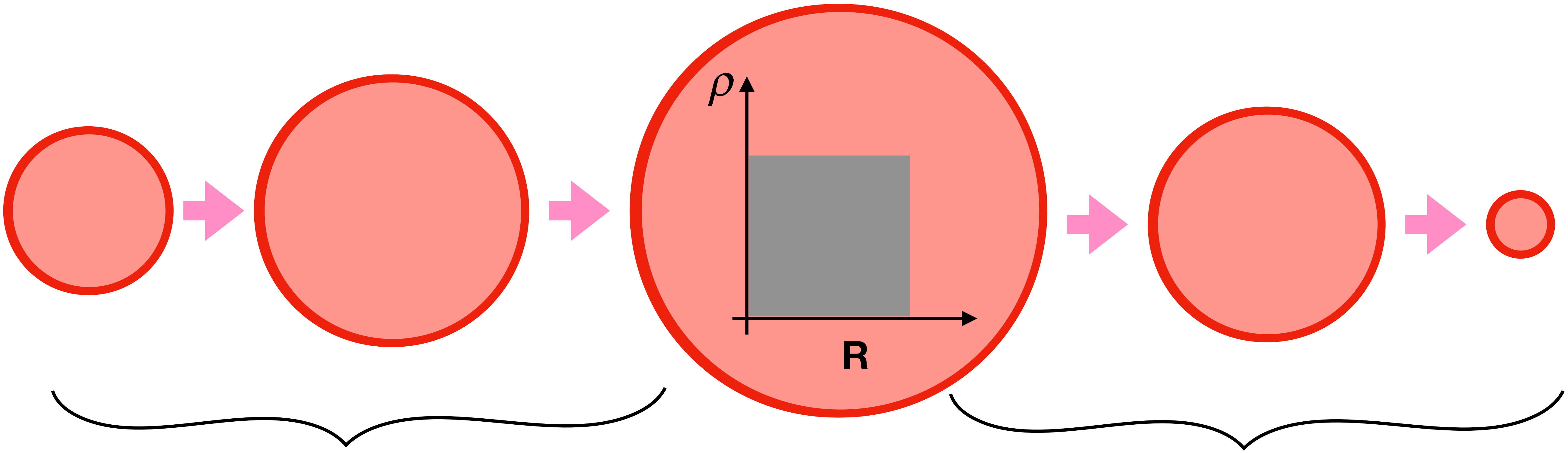
Hence, higher-order moments are required to completely specify density field.

How to proceed?

- Oversimplified, but insightful, analytical model (this lecture)
- Higher-order perturbation theory (see MBW §4.1.7)
- Numerical simulations (see MBW §5.6.2)
- The Halo Model (see MBW §7.6)

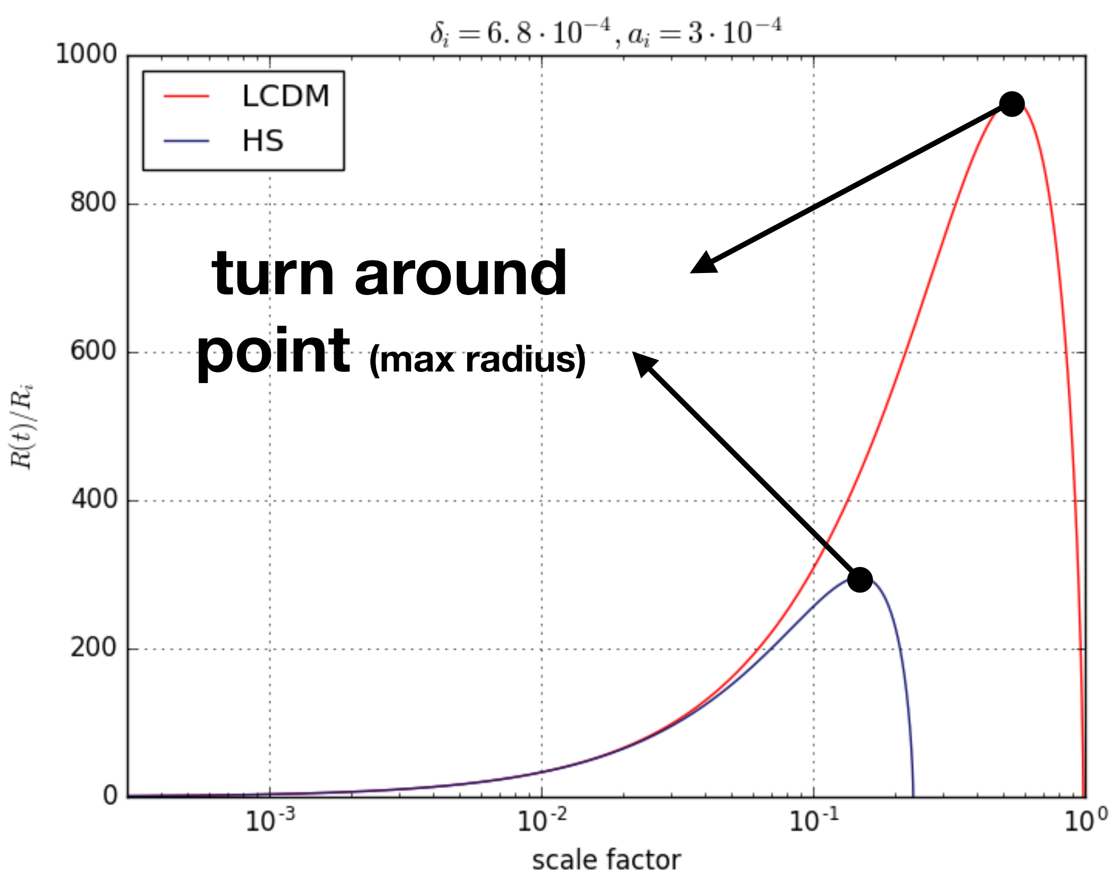
[from F. Bosch]

Halo formation—Spherical Collapse



expand with background

decouple from background expansion



$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = \varepsilon$$

assuming initially, shell is co-moving

$$\dot{r} = Hr$$

One can prove that, in EdS universe,
for arbitrary over density regime, it will always collapse!

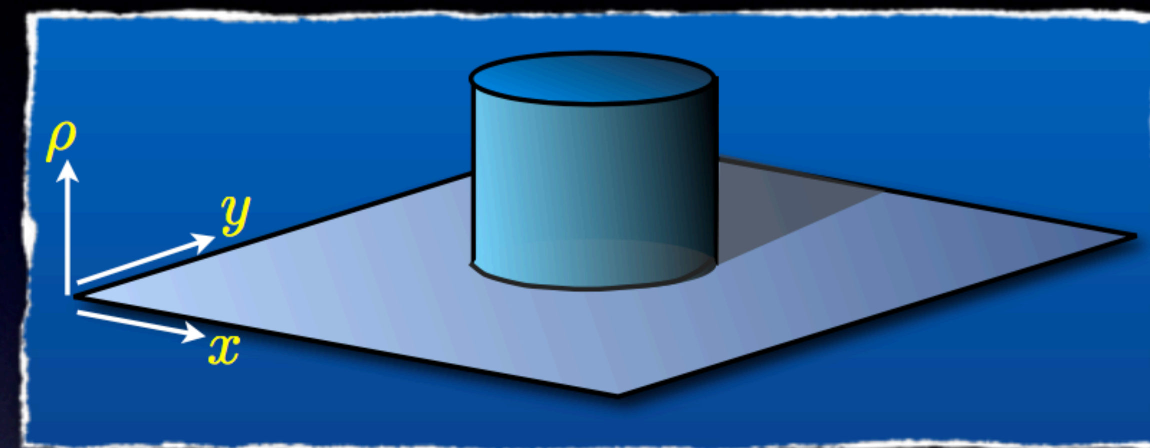
early collapse ~ higher density; later collapse ~ lower density

$$\varepsilon = -\frac{1}{2} H_i^{\,2} r_i^{\,2} \delta_i < 0$$

Top-Hat Spherical Collapse

In order to gain insight into the non-linear evolution of density perturbations, we now consider the highly idealized case of Top-Hat Spherical Collapse.

- Universe is homogeneous, except for a single, top-hat, spherical perturbation.
- Universe is in matter-dominated phase, after recombination...
- Collisionless fluid → treatment is only valid for collisionless Dark Matter.
- Einstein-de Sitter (EdS) cosmology



$$\Omega_{\rm m}(t) = 1 \qquad H(t)\cdot t = \frac{2}{3}$$

$$\bar{\rho} = \frac{1}{6\pi G t^2} \qquad D(a) = a \propto t^{2/3}$$

NOTE:

Although the following treatment is only valid for an EdS cosmology, similar models can be constructed for other cosmologies as well, including ΛCDM (see MBW §5.1.1 + 5.1.2)

Furthermore, since all cosmologies behave similar to EdS at early times, this treatment is always good approximation at high z....

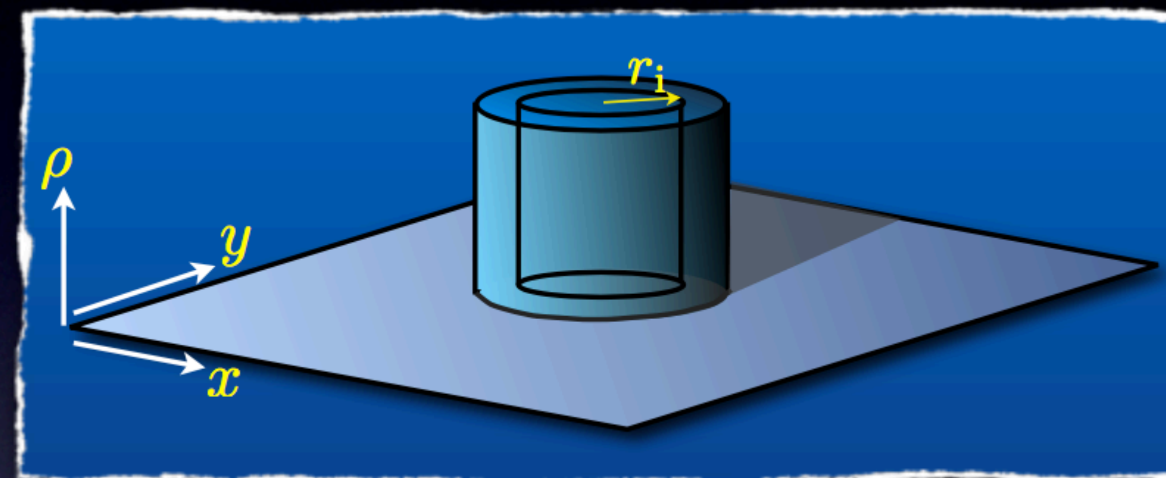
[from F. Bosch]

Top-Hat Spherical Collapse

Consider our spherical top-hat perturbation: Let $r_{\rm i}$ denote the radius of some mass shell inside the top-hat at some initial time, $t_{\rm i}$, and let $\delta_{\rm i}$ and $\bar{\rho}_{\rm i}$ denote the top-hat overdensity and the back-ground density at that same time.

The mass enclosed by the shell is

$$\begin{aligned} M(<r) &= \frac{4}{3}\pi r_{\rm i}^3\,\bar{\rho}_{\rm i}\,[1+\delta_{\rm i}] \\ &= \frac{4}{3}\pi r^3(t)\,\bar{\rho}(t)\,[1+\delta(t)] \end{aligned}$$



where the second equality expresses mass conservation: because of spherical symmetry, the mass inside the shell is conserved, but only up to shell crossing !!!

Newton's first Theorem:

a spherically symmetric matter distribution outside a sphere exerts no force on that sphere

Equation of motion

$$\frac{{\rm d}^2 r}{{\rm d}t^2} = -\frac{GM}{r^2}$$

r(t) is the Lagrangian coordinate

[from F. Bosch]

Top-Hat Spherical Collapse

Integrating the equation of motion once yields $\boxed{\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{GM}{r} = E}$

where the integration constant E is clearly the specific energy of our shell.

For $E < 0$, mass shell is bound, and solution can be written in following parametric form:

$$r = A\,(1 - \cos\theta) \qquad \theta \in [0, 2\pi]$$
$$t = B\,(\theta - \sin\theta)$$
$$A = \frac{GM}{2|E|} \qquad B = \frac{GM}{(2|E|)^{3/2}} \quad \Rightarrow \quad A^3 = GMB^2$$

This solution implies the following evolution for our mass shell:

- shell expands from $r = 0$ at $\theta = 0$ $(t = 0)$
- shell reaches a maximum radius $r_{\rm max}$ at $\theta = \pi$ $(t = t_{\rm max} = \pi B)$
- shell collapses back to $r = 0$ at $\theta = 2\pi$ $(t = t_{\rm coll} = 2t_{\rm max})$

The time of maximum size is often called the turn-around time, $t_{\rm ta} = t_{\rm max}$, while the time of collapse is also called the virialization time $t_{\rm vir} = t_{\rm coll} = 2t_{\rm ta}$

[from F. Bosch]

Top-Hat Spherical Collapse

Now let us focus on the evolution of the actual overdensity:

The mean density of the top-hat is $\rho = \frac{3M}{4\pi r^3} = \frac{3M}{4\pi A^3}(1-\cos\theta)^{-3}$

The mean density of the background is $\bar{\rho} = \frac{1}{6\pi G t^2} = \frac{1}{6\pi G B^2}(\theta - \sin\theta)^{-2}$

Hence, the actual overdensity of our spherical top-hat region, according to the spherical collapse (SC) model, which in general will be non-linear, is

$$\boxed{1+\delta = \frac{\rho}{\bar{\rho}} = \frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3}}$$

where we have used that $A^3 = GMB^2$.

Before we examine this SC model in some detail, we first compare it to predictions from linear theory....

[from F. Bosch]

Top-Hat Spherical Collapse

For a number of reasons (in particular for use in EPS theory), it is also useful to compare this SC overdensity model to what linear theory predicts for $\delta(t)$.

According to linear theory, perturbation in EdS cosmology evolve as

$$\boxed{\delta_{\rm lin} \propto D(a) \propto a \propto t^{2/3}}$$

In order to use the correct initial conditions (ICs), we have to use our parametric solution of $r(t)$ in the limit $\theta \ll 1$. Using a Taylor series expansion of $\sin\theta$ and $\cos\theta$ one can show that:

$$\boxed{\delta_{\rm i} = \frac{3}{20}(6\pi)^{2/3}\left(\frac{t_{\rm i}}{t_{\rm max}}\right)^{2/3} \qquad (\delta_{\rm i} \ll 1)}$$

NOTE: this implies that since $\delta(r) =$ constant inside the top-hat, each mass shell that is part of the top-hat will turn-around (reach maximum expansion) at the same time....

[from F. Bosch]

Top-Hat Spherical Collapse

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Combining the above, we have that, according to linear theory:

$$\boxed{\delta_{\rm lin} = \delta_{\rm i}\left(\frac{t}{t_{\rm i}}\right)^{2/3} = \frac{3}{20}(6\pi)^{2/3}\left(\frac{t}{t_{\rm max}}\right)^{2/3}}$$

[from F. Bosch]

Turn-Around & Collapse

Spherical Collapse (SC) model:

$$1+\delta = \frac{\rho}{\bar{\rho}} = \frac{9}{2}\,\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3}$$

Linear Theory

$$\delta_{\rm lin} = \delta_{\rm i}\left(\frac{t}{t_{\rm i}}\right)^{2/3} = \frac{3}{20}(6\pi)^{2/3}\left(\frac{t}{t_{\rm max}}\right)^{2/3}$$

Turn-Around: $(t_{\rm ta} = t_{\rm max};\, \theta = \pi)$

SC model: $1+\delta(t_{\rm ta}) = \dfrac{9\pi^2}{16} \simeq 5.55$

linear theory: $\delta_{\rm lin}(t_{\rm ta}) = \dfrac{3}{20}(6\pi)^{2/3} \simeq 1.062$

Collapse (shell crossing) $(t_{\rm coll} = 2t_{\rm ta})$

SC model: $\delta(t_{\rm coll}) = \infty$

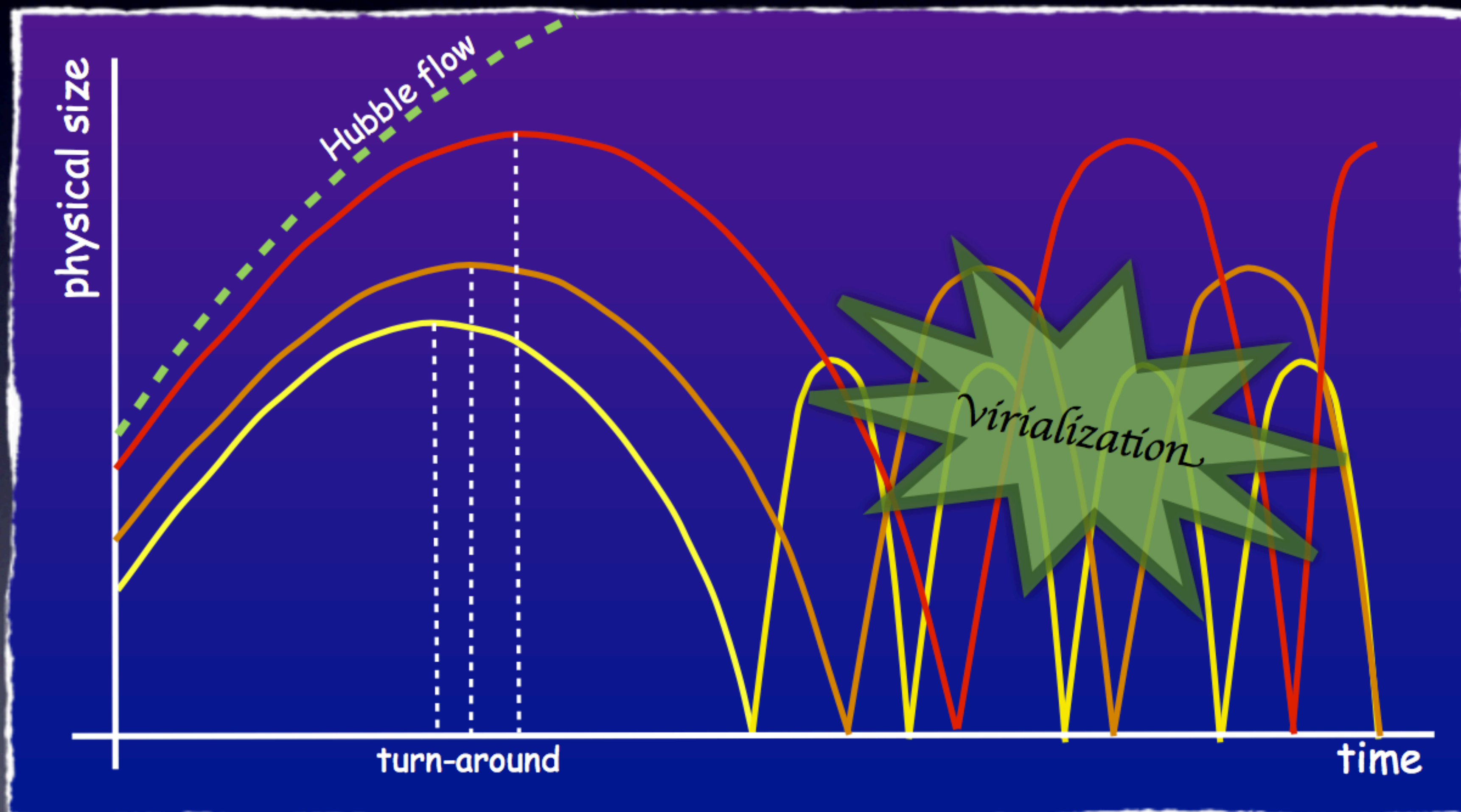
linear theory: $\delta(t_{\rm coll}) = \dfrac{3}{20}(12\pi)^{2/3} = \dfrac{3}{5}\left(\dfrac{3\pi}{2}\right)^{2/3} \simeq 1.686$

[from F. Bosch]

Shell Crossing & Virialization

The SC model discussed above is only valid up to the point of shell crossing. Afterall, after shell crossing M(r) is no longer a conserved quantity!

According to the SC model, $\delta(t_{\rm coll}) = \infty$, which would result in the formation of a black hole. However, in reality, the collapse is never perfectly spherical.



Individual oscillating shells interact gravitationally, exchanging energy (virializing). This process, to be described in more detail below, results in a virialized dark matter halo

[from F. Bosch]

Final Density of a Collapsed Dark Matter Halo

Virialization means that the system relaxes towards virial equilibrium:

We can use the virial theorem to make a simple estimate of the final density of our collapsed & virialized dark matter halo:

Virial Equilibrium:	$2\,K_{\rm f} + W_{\rm f} = 0$
Energy conservation:	$E_{\rm f} = K_{\rm f} + W_{\rm f} = E_{\rm i} = E_{\rm ta}$

$$\left.\begin{aligned} E_{\rm ta} &= W_{\rm ta} = -\frac{GM}{r_{\rm ta}} \\ E_{\rm f} &= W_{\rm f}/2 = -\frac{GM}{2r_{\rm vir}} \end{aligned}\right\} \Rightarrow \boxed{r_{\rm vir} = r_{\rm ta}/2}$$

A mass shell is expected to virialize at half its turn-around radius.

Hence, after virialization, the average density of the material enclosed by the mass shell is 8 times denser than at turn-around....

[from F. Bosch]

向心力 $$\frac{mv^2}{r} = \frac{GMm}{r^2}$$ 引力

动能 $$\frac{mv^2}{2}$$ $$\frac{-GmM}{r}$$ 势能

Final Density of a Collapsed Dark Matter Halo

We now compute the average overdensity of a virialized dark matter halo:

$$\boxed{1 + \Delta_{\rm vir} \equiv 1 + \delta(t_{\rm coll}) = \frac{\rho(t_{\rm coll})}{\bar{\rho}(t_{\rm coll})}}$$

NOTE: for consistency with many textbooks and journal articles, we use the symbol $\Delta_{\rm vir}$, rather than $\delta_{\rm vir}$ to indicate the virialized overdensity....

Using that $\bar{\rho} \propto a^{-3} \propto t^{-2}$ (EdS), and that $t_{\rm coll} = 2t_{\rm ta}$ we have that

$$\boxed{1 + \Delta_{\rm vir} = \frac{8\,\rho_{\rm ta}}{\bar{\rho}(t_{\rm ta})/4} = 32\,(1+\delta_{\rm ta}) = 18\pi^2 \simeq 178}$$

For non-EdS cosmologies, the virial overdensities are well approximated by

$$\Delta_{\rm vir} \approx (18\pi^2 + 60\,x - 32\,x^2)/\Omega_{\rm m}(t_{\rm vir}) \qquad (\Omega_\Lambda = 0)$$

$$\Delta_{\rm vir} \approx (18\pi^2 + 82\,x - 39\,x^2)/\Omega_{\rm m}(t_{\rm vir}) \qquad (\Omega_\Lambda \neq 0)$$

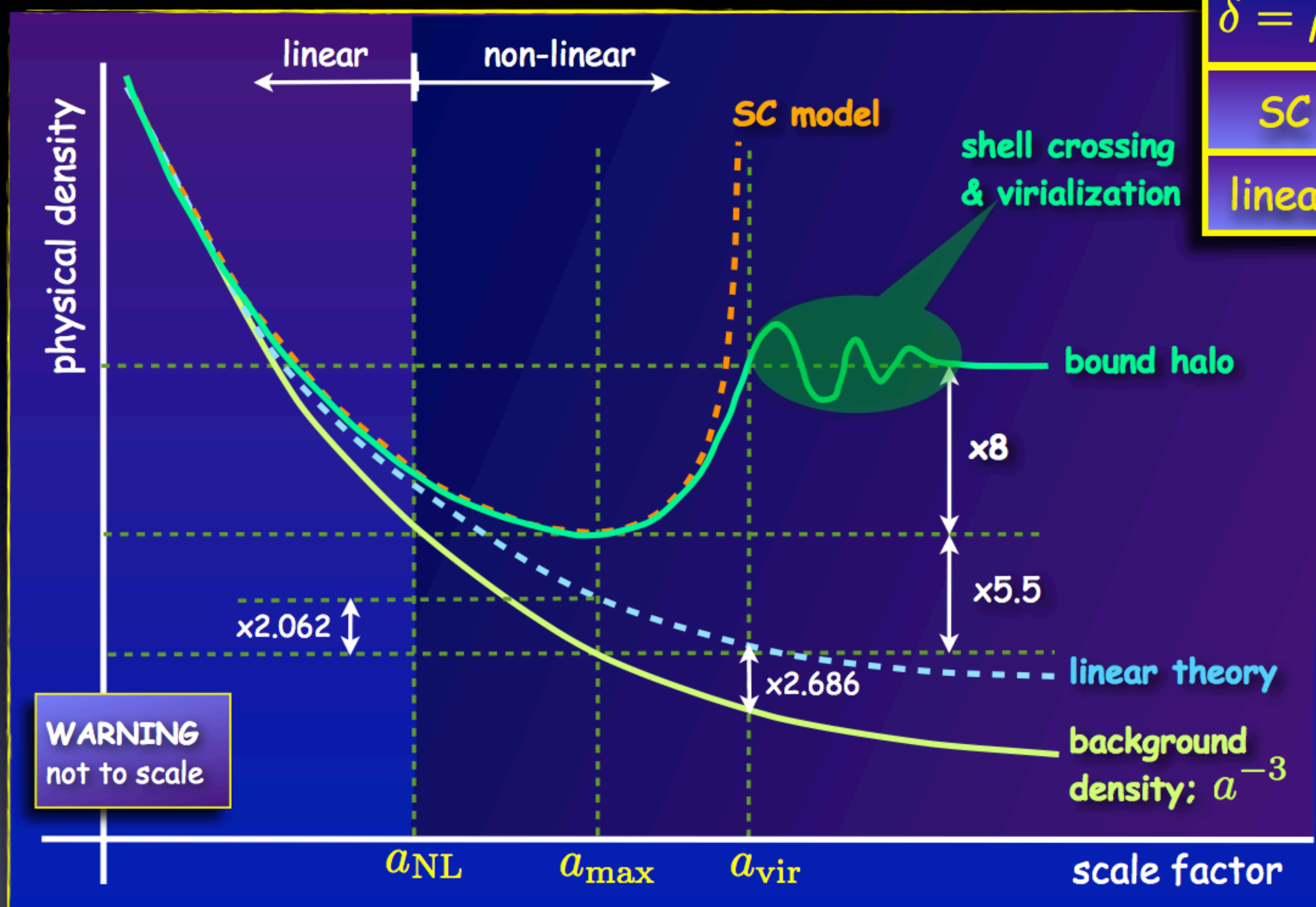
(Bryan & Norman 1998)

Here $x = \Omega_{\rm m}(t_{\rm vir}) - 1$. These equations are often used to `define' dark matter haloes in N-body simulations or in analytical models....

[from F. Bosch]

Summary: The Spherical Collapse (SC) Model

$\delta = \rho/\bar{\rho} - 1$	turn-around	collapse
SC model	4.55	∞
linear model	1.062	1.686



Although SC model becomes inaccurate (brakes down) shortly after turn-around it is still useful for identifying important epochs in linearly evolved density field...

The linearly extrapolated density field collapses when $\delta_{\rm lin} = \delta_c \simeq 1.686$

Virialized dark matter haloes have an average overdensity of $\Delta_{\rm vir} \simeq 178$

[from F. Bosch]

Relaxation & Virialization

More details in MBW §5.4

Relaxation: the process by which a physical system acquires equilibrium or returns to equilibrium after a disturbance. Often, but not always, relaxation erases the system's "knowledge" of it's initial conditions.

Virialization: the process by which a physical system settles in virial equilibrium

Virial Equilibrium: A system is said to be in virial equilibrium if

$$\boxed{2K + W + \Sigma = 0}$$

K = kinetic energy
W = potential energy
Σ = work done by surface pressure

Often, Σ can be ignored, in which case virial equilibrium implies that $E = -K = W/2$

Two-body relaxation time: the time required for a particle to change its kinetic energy by about its initial amount due to two-body interactions

As you learn in Galactic Dynamics, the two-body relaxation time, $\boxed{t_{\rm relax} \simeq \frac{N}{10 \ln N} t_{\rm cross}}$

Here N is the number of particles and $t_{\rm cross} \sim R/v$ is the system's crossing time.

For almost all collisionless systems of interest to us (galaxies, dark matter haloes) it is easy to show that $t_{\rm relax} \gg t_{\rm Hubble} \simeq 1/H_0$

[from F. Bosch]

The Zel'dovich Approximation

So far we considered perturbations in Eulerian (`grid') coordinates. Individual overdensities stay at a fixed (comoving) position and grow or decay in amplitude....

We now switch to Lagrangian description, which follows motion of individual particles. This gives insights into dynamics of structure formation process, and, unlike its Eulerian counterpart, remains (fairly) accurate in the mildly non-linear regime...

It is easy to see that Eulerian description brakes down in mildly non-linear regime: Once overdensities ($\delta_{\rm i} > 0$) reach amplitudes of order unity, the underdensities ($\delta_{\rm i} < 0$) have grown to $\delta < -1$, which would imply a negative (=unphysical) density...

Zel'dovich (1970) came up with a Lagrangian formalism that is based on the following approximation (known as Zel'dovich Approximation, ZA):

particles continue to move in the direction of their initial displacement

$$\Rightarrow \quad \vec{x}(t) = \vec{x}_{\rm i} - c(t) \cdot \vec{f}(\vec{x}_{\rm i})$$

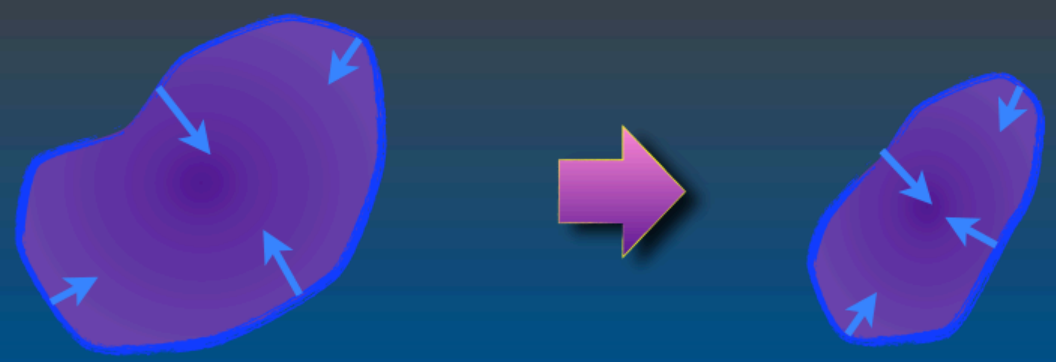
- $\vec{x}_{\rm i}$ initial (Lagrangian), comoving coordinates
- $c(t)$ function of time, to be determined below
- $\vec{f}(\vec{x}_{\rm i})$ vector function of initial coordinates, specifying direction of velocity

[from F. Bosch]

The Zel'dovich Approximation

$$\boxed{\textbf{ZA:}\ \vec{x}(t) = \vec{x}_\mathrm{i} - c(t) \cdot \vec{f}(\vec{x}_\mathrm{i})}$$

Note: the ZA is exact if perturbation is a 1D sheet in an otherwise homogeneous universe; in that case direction of velocity remains fixed...



Mass conservation

$$\rho(\vec{x}, t)\, a^3(t)\, \mathrm{d}^3\vec{x} = \rho_\mathrm{i}(\vec{x}_\mathrm{i})\, \mathrm{d}^3\vec{x}_\mathrm{i}$$

Here $a(t)$ is the scale-factor normalized to unity at the initial time t_i: the scaling with $a^3(t)$ is required since $\vec{x}$ are comoving coordinates. The equation of mass conservation is valid (up to orbit crossing) for any geometry; no spherical symmetry is required!!

Using Linear Algebra: $$\boxed{\rho(\vec{x}, t) = \rho_\mathrm{i}(\vec{x}_\mathrm{i})\, a^{-3} \left\| \frac{\mathrm{d}\vec{x}}{\mathrm{d}\vec{x}_\mathrm{i}} \right\|^{-1}}$$

Here $\|A\| = \det(A) = \prod_i \lambda_i$ with λ_i the eigenvectors of the matrix A

[from F. Bosch]

The Zel'dovich Approximation

Using that the tensor $\left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}\vec{x}_\mathrm{i}}\right)_{jk} = \delta_{jk} - c(t)\frac{\partial f_j}{\partial x_k}$ we have that

$$\boxed{\rho(\vec{x},t) = \rho_\mathrm{i}(\vec{x}_\mathrm{i})\, a^{-3} \frac{1}{(1-c\lambda_1)\,(1-c\lambda_2)\,(1-c\lambda_3)}}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are the eigenvalues of the deformation tensor $\partial f_i / \partial x_j$

Using that $\rho_\mathrm{i}(\vec{x}_\mathrm{i}) = \bar{\rho}_\mathrm{i}\,[1+\delta_\mathrm{i}(\vec{x}_\mathrm{i})] \simeq \bar{\rho}_\mathrm{i}$ and that $\bar{\rho}(t)\,a^3 = \bar{\rho}_\mathrm{i}\,a_\mathrm{i}^3$ this yields

$$\boxed{1+\delta(\vec{x},t) = \frac{\rho(\vec{x},t)}{\bar{\rho}(t)} = \frac{1}{(1-c\lambda_1)\,(1-c\lambda_2)\,(1-c\lambda_3)}}$$

(recall that $a_i = 1$)

We can gain some useful insight from this equation (using that $c(t) > 0$) :

- if $\lambda_i > 0$ this implies collapse in the direction of the i^{th} eigenvector.
- if $\lambda_i < 0$ this implies expansion in the direction of the i^{th} eigenvector.
- if $c(t) = 1/\lambda_i$ `shell' crossing happens along the direction of the i^{th} eigenvector.
- as long as $c\lambda_1 \ll 1$ the perturbation is still in the linear regime.

[from F. Bosch]

The Zel'dovich Approximation

Linearization of the equation for the density perturbation yields

$$\boxed{1+\delta(\vec{x},t) = \frac{1}{1-c\,(\lambda_1+\lambda_2+\lambda_3)} \simeq 1 + c\,(\lambda_1+\lambda_2+\lambda_3)}$$

Hence, we have that, in the linear regime $\delta(\vec{x},t) = c(t)\,\mathrm{Tr}(\vec{\nabla}\cdot\vec{f}) = c(t)\,\vec{\nabla}\cdot\vec{f}$
where we have used that the divergence of a vector field is a scalar.

If we compare this to the fact that, in the linear regime, $\delta(\vec{x},t) = D(t)\,\delta_{\rm i}(\vec{x}_{\rm i})$
we see that $c(t) = D(t)$ and $\vec{\nabla}\cdot\vec{f} = \delta_{\rm i}$.

Using the Poisson equation, according to which $\delta_{\rm i} = \nabla^2\Phi_{\rm i}/4\pi G\bar{\rho}_{\rm i}$ (recall that $a_{\rm i}=1$)
and the fact that $\nabla^2\Phi = \vec{\nabla}\cdot\vec{\nabla}\Phi$, we finally see that $\vec{f} = \vec{\nabla}\Phi_{\rm i}/4\pi G\bar{\rho}_{\rm i}$

$$\boxed{\vec{x}(t) = \vec{x}_{\rm i} - \frac{D(a)}{4\pi G\bar{\rho}_{\rm i}}\,\vec{\nabla}\Phi_{\rm i}}$$

Zel'dovich Approximation

[from F. Bosch]

Zel'dovich Pancakes

The ZA describes the non-linear evolution of density perturbations. It has two important advantages over the spherical collapse model:

- it makes no oversimplified assumptions about geometry
- it remains accurate well into the quasi-linear regime

To understand why the ZA is more accurate in the quasi-linear regime (brakes down at a later stage), have a look at its predicted evolution for an overdensity:

$$\boxed{1+\delta(\vec{x},t) = \frac{\rho(\vec{x},t)}{\bar{\rho}(t)} = \frac{1}{(1-c\lambda_1)\,(1-c\lambda_2)\,(1-c\lambda_3)}}$$

It is clear from this equation that collapse happens first along the axis associated with the first (largest) eigenvalue, λ_1 ⇒ gravity accentuates asphericity!

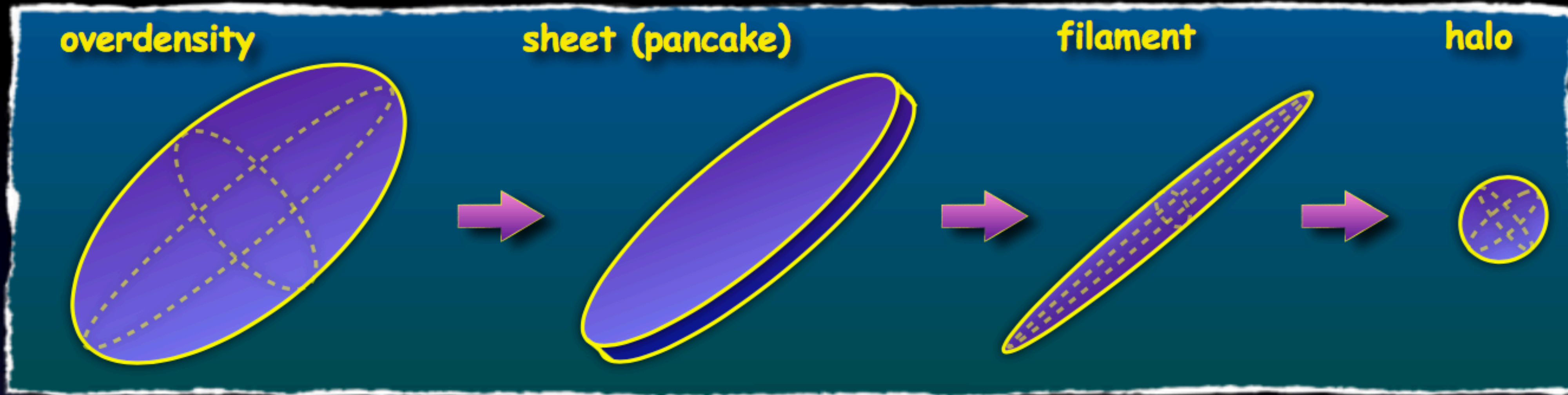
Hence, collapse leads to flattened structures, called (Zel'dovich) pancakes. The ZA approximation is so accurate simply because, as mentioned above, it becomes exact in the limit of planar perturbations...

Because ZA is so accurate, it is often used in setting up the initial conditions for N-body simulations..

[from F. Bosch]

Ellipsoidal Collapse

As is evident from the ZA, in general density perturbations will collapse according to:



For a uniform, ellipsoidal overdensity in homogeneous universe (ellipsoidal top-hat) one can obtain analytical approximations for time evolution of its 3 principal axes (see MBW §5.3).

This can be used to compute the critical overdensity for collapse (of the longest axis = `halo formation') in linear theory. The result can be obtained by solving

$$\boxed{\frac{\delta_{\rm ec}}{\delta_{\rm sc}} \approx 1 + 0.47 \left[5(e^2 \pm p^2)\,\frac{\delta_{\rm ec}^2}{\delta_{\rm sc}^2}\right]^{0.615}}$$

Sheth, Mo & Tormen (2001)

Here $\delta_{\rm ec} = \delta_{\rm ec}(e,p)$ is the critical overdensity for ellipsoidal collapse, $\delta_{\rm sc} = \delta_{\rm c} \simeq 1.686$ is the critical overdensity for spherical collapse, and the plus (minus) sign is used if p is negative (positive)....

[from F. Bosch]

Ellipsoidal Collapse

$$\frac{\delta_{\rm ec}}{\delta_{\rm sc}} \approx 1 + 0.47 \left[5(e^2 \pm p^2) \frac{\delta_{\rm ec}^2}{\delta_{\rm sc}^2} \right]^{0.615}$$

Ellipsoidal collapse

The parameters e and p characterize the asymmetry of the initial tidal field:

$$e \equiv \frac{\lambda_1 - \lambda_3}{2(\lambda_1 + \lambda_2 + \lambda_3)} \qquad p \equiv \frac{\lambda_1 + \lambda_3 - 2\lambda_2}{2(\lambda_1 + \lambda_2 + \lambda_3)}$$

Note that for a spherical system $\lambda_1 = \lambda_2 = \lambda_3$ $\rightarrow$ $e = p = 0$ $\rightarrow$ $\delta_{\rm ec} = \delta_{\rm sc} \simeq 1.686$

In general, however, $\lambda_1 > \lambda_2 > \lambda_3$ which results in $\delta_{\rm ec} > \delta_{\rm sc}$, which implies that structures collapse later under ellipsoidal collapse conditions (more realistic) than under spherical collapse conditions.

As a final remark, as we will see later, less massive structures are more strongly influenced by tides and therefore more ellipsoidal...This has important implications....

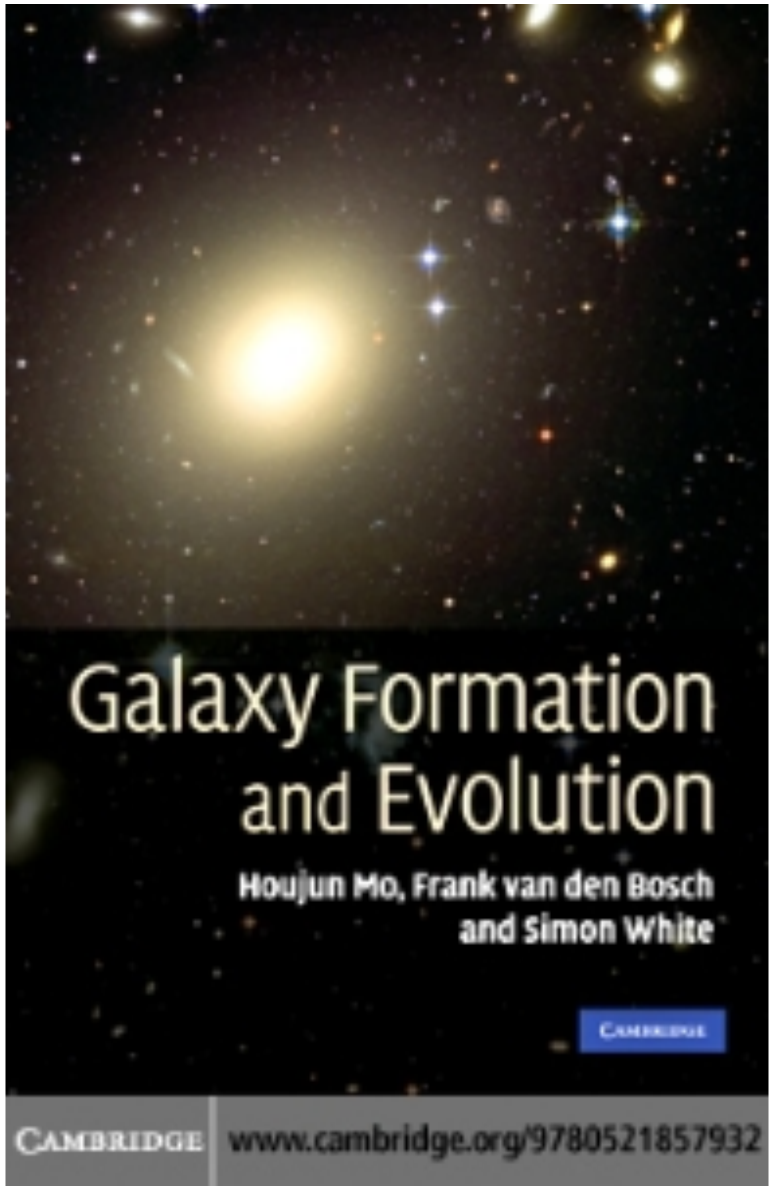
[from F. Bosch]

FurtherReading

Frank van den Bosch Lecture 1~4

http://www.as.huji.ac.il/schools/phys30/media

Chapter 5.1



Hou-Jun Mo @Tsinghua

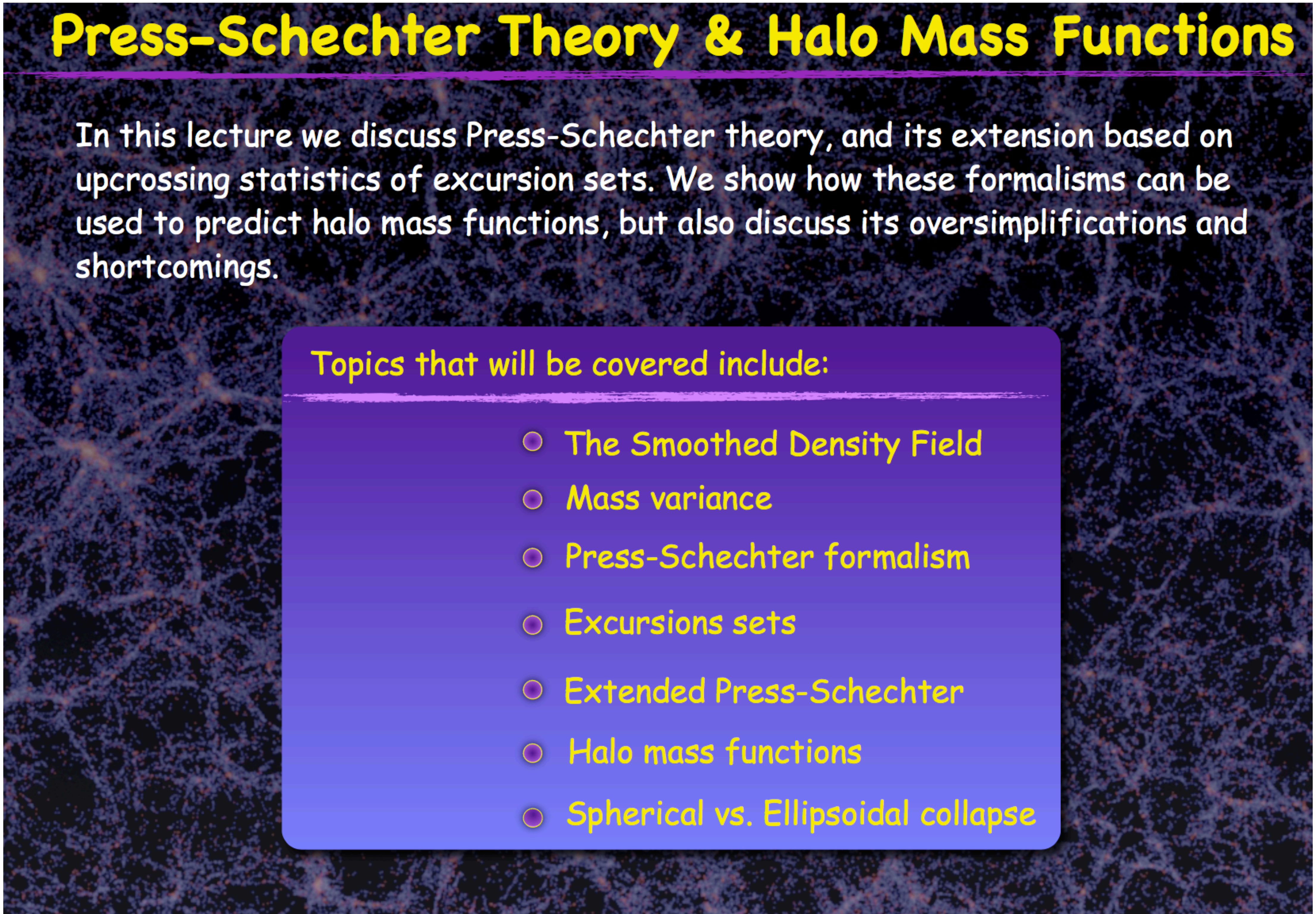
F. Bosch @Yale

Simon White @MPA

2. Halo number density per Mass Bin

Now, we understand how a SINGLE non-linear structure (Halo) formed.

Our next task is to study the spatial distribution of those halos!



[from F. Bosch]

Smoothing

Given a density field $\delta(\vec{x})$, one can filter it using some window function (or "filter") $W(\vec{x};R)$ which is properly normalized such that $\int W(\vec{x};R)\,\mathrm{d}^3\vec{x} = 1$, to get a smoothed field

$$\boxed{\delta(\vec{x};R) \equiv \int \delta(\vec{x}')\,W(\vec{x}-\vec{x}';R)\,\mathrm{d}^3\vec{x}'}$$

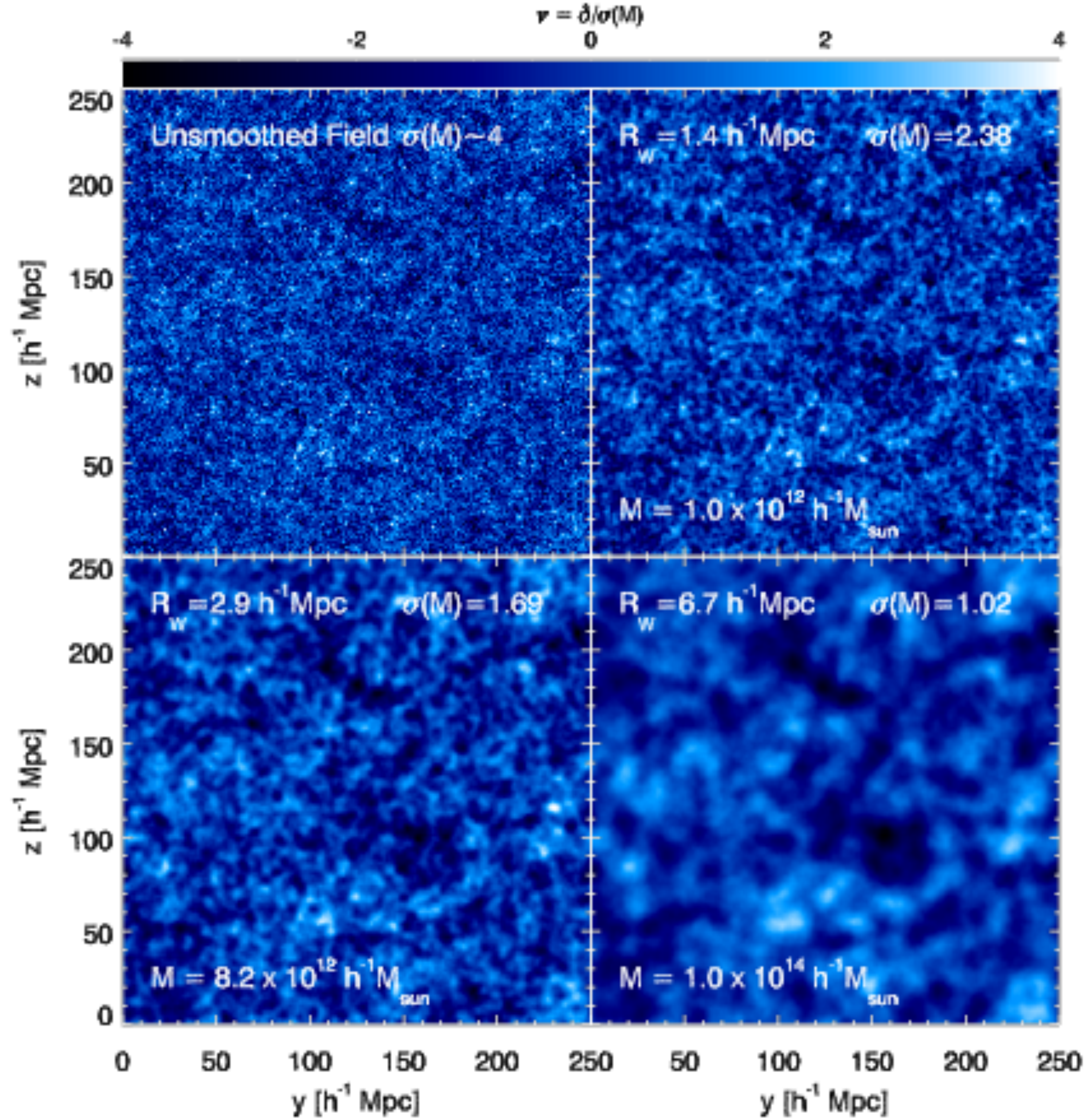
For each filter, one can define a mass $M = \gamma_{\rm f}\,\bar{\rho}\,R^3$, where $\gamma_{\rm f}$ is some constant that depends on the shape of the filter. In what follows, we will characterize a filter intermittendly by its size R or its mass M.

The above equation for the smoothed density field is a convolution integral (the density field is convolved with the window function). Since convolution in real-space is equal to multiplication in Fourier space, we have that

$$\boxed{\delta(\vec{k};R) = \int \delta(\vec{x};R)\,e^{-i\vec{k}\cdot\vec{x}}\,\mathrm{d}^3\vec{x} = \delta(\vec{k})\,\widetilde{W}(kR)}$$

where $\widetilde{W}(kR) = \int W(\vec{x};R)\,e^{-i\vec{k}\cdot\vec{x}}\,\mathrm{d}^3\vec{x}$ is the Fourier Transform of the window function for which we have made it explicit that k and R only enter in the combination kR.

[from F. Bosch]



ν = δ/σ(M)
-4
-2
0
2
4
Unsmoothed Field σ(M)~4
R_W=1.4 h^-1Mpc
σ(M)=2.38
M = 1.0 x 10^12 h^-1M_sun
R_W=2.9 h^-1Mpc
σ(M)=1.69
M = 8.2 x 10^12 h^-1M_sun
R_W=6.7 h^-1Mpc
σ(M)=1.02
M = 1.0 x 10^14 h^-1M_sun
z [h^-1 Mpc]
y [h^-1 Mpc]
0
50
100
150
200
250

Window Functions

Throughout we will use either one of the following three window functions:

Top Hat Filter: $\gamma_f = 4\pi/3$

$$W(\vec{x};R) = \begin{cases} \frac{3}{4\pi R^3} & r \leq R \\ 0 & r > R \end{cases} \qquad \widetilde{W}(kR) = \frac{3}{(kR)^3}\left[\sin(kR) - (kR)\cos(kR)\right]$$

Gaussian Filter: $\gamma_f = (2\pi)^{3/2}$

$$W(\vec{x};R) = \frac{1}{(2\pi)^{3/2} R^3} \exp\left(-\frac{r^2}{2R^2}\right) \qquad \widetilde{W}(kR) = \exp\left(-\frac{(kR)^2}{2}\right)$$

Sharp k-space Filter: $\gamma_f = 6\pi^2$

$$W(\vec{x};R) = \frac{1}{2\pi^2 r^3}\left[\sin(r/R) - (r/R)\cos(r/R)\right] \qquad \widetilde{W}(kR) = \begin{cases} 1 & k \leq 1/R \\ 0 & k > 1/R \end{cases}$$

[from F. Bosch]

Un-smoothed density field

The Variance of the Density Field

Recall that the assumption of ergodicity implies that $\langle\delta\rangle = \frac{1}{V}\int \delta(\vec{x})\,\mathrm{d}^3\vec{x}$

where V is the volume of the Universe over which we assume it to be periodic.

Similarly, we have that the variance of the density field can be written as

$$\boxed{\sigma^2 = \langle\delta^2\rangle = \frac{1}{V}\int \delta^2(\vec{x})\,\mathrm{d}^3\vec{x}}$$

Recall that $\xi(r) = \langle\delta(\vec{x})\delta(\vec{x}+\vec{r})\rangle = \frac{1}{(2\pi)^3}\int P(k)e^{+i\vec{k}\cdot\vec{r}}\mathrm{d}^3\vec{k}$, from which it is clear that

$$\boxed{\sigma^2 = \xi(0) = \frac{1}{(2\pi)^3}\int P(k)\mathrm{d}^3\vec{k} = \frac{1}{2\pi^2}\int P(k)\,k^2\,\mathrm{d}k = \int \Delta^2(k)\frac{\mathrm{d}k}{k}}$$

where $\Delta^2(k) = \frac{k^3}{2\pi^2}P(k)$ is the unitless power spectrum.

[from F. Bosch]

Smoothed density field

The Smoothed Density Field

Similar to case without smoothing, we define the variance of the smoothed density field as

$$\sigma^2(R) = \langle \delta^2(\vec{x}; R) \rangle = \frac{1}{2\pi^2} \int P(k)\, \widetilde{W}^2(kR)\, k^2\, \mathrm{d}k$$

Note that $\lim_{R\to 0} \widetilde{W}(kR) = 1$ (normalization condition), from which it is clear that $\lim_{R\to 0} \sigma^2(R) = \sigma^2$ as required.

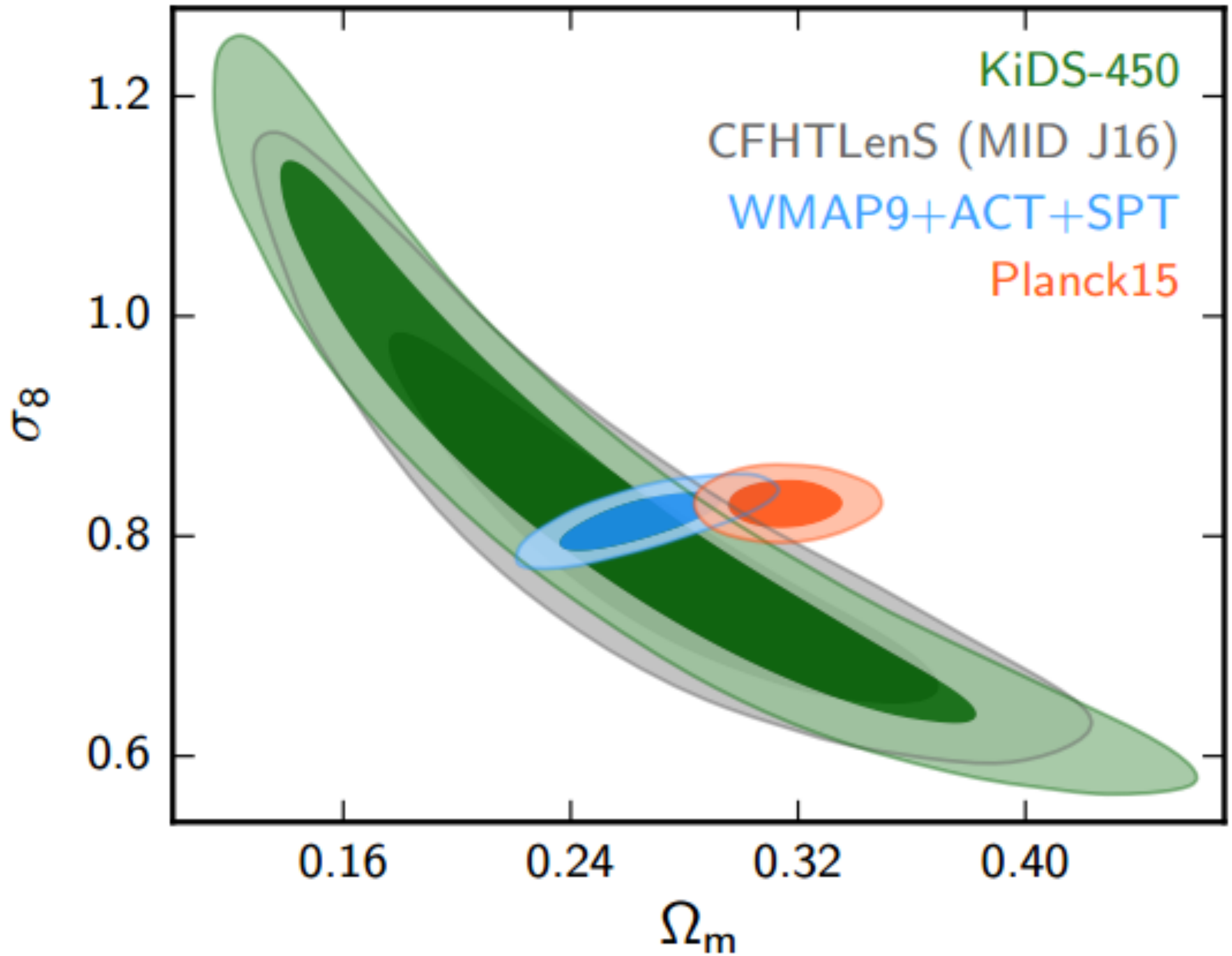
The cosmological parameter σ_8 is defined as the variance of the density field, linearly extrapolated to $z = 0$, when smoothed with top-hat filter of size $R = 8h^{-1}\mathrm{Mpc}$

$$\sigma_8 = \langle \delta^2_{\mathrm{lin}}(\vec{x}; R) \rangle^{1/2} = \left[\frac{1}{2\pi^2} \int P_{\mathrm{lin}}(k)\, \widetilde{W}^2_{\mathrm{TH}}(kR)\, k^2\, \mathrm{d}k \right]^{1/2}$$

This parameter is used to characterize the normalization of the power spectrum. It's currently favored value is of the order of $\sigma_8 \simeq 0.8 \pm 0.1$. A larger value of σ_8 implies larger fluctuations, and therefore earlier structure formation...

[from F. Bosch]

There are some tensions in S8!



Mass Variance

Since we can equally label a filter by its size R or its mass M, we can write $\sigma^2(R) = \sigma^2(M)$. The latter is called the mass variance, and plays an important role in what follows.

It is straightforward to show that

$$\boxed{\sigma^2(M) = \left\langle \left(\frac{M(\vec{x};R) - \bar{M}(R)}{\bar{M}(R)} \right)^2 \right\rangle}$$

where $M(\vec{x};R) = V_R \int \rho(\vec{x}')\, W(\vec{x}-\vec{x}';R)\, \mathrm{d}^3\vec{x}'$, with V_R the volume of the filter, and $\bar{M}(R) = \langle M(\vec{x};R) \rangle$, which exemplifies the nomenclature \`mass variance'.

NOTE: If $\delta(\vec{x})$ is a Gaussian random field, then so is $\delta(\vec{x};R)$. In particular

$$\boxed{\mathcal{P}(\delta_M)\,\mathrm{d}\delta_M = \frac{1}{\sqrt{2\pi}\,\sigma_M} \exp\left[-\frac{\delta_M^2}{2\sigma_M^2} \right] \mathrm{d}\delta_M}$$

where we have used the shorthand notation $\delta_M = \delta(\vec{x};M)$ and $\sigma_M = \sigma(M)$.

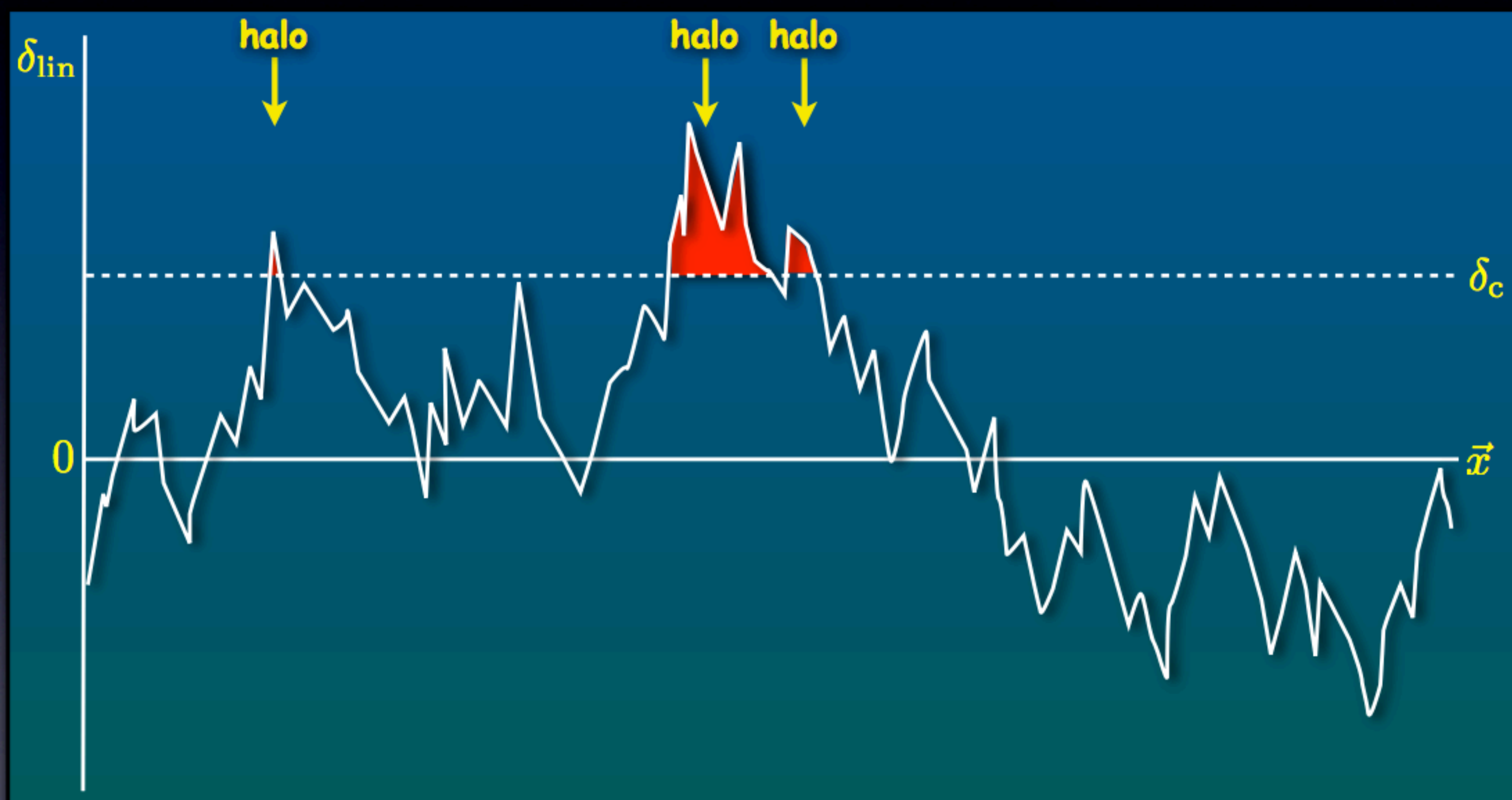
[from F. Bosch]

peak background split approach

The Linear Cosmological Density Field

According to linear theory, the density field evolves as $\delta(\vec{x},t) = D(t)\,\delta_0(\vec{x})$

Here $\delta_0(\vec{x})$ is the density field linearly extrapolated to $t = t_0$, and $D(t)$ is the linear growth rate normalized to unity at $t = t_0$



According to the spherical collapse model, regions with $\delta(\vec{x},t) > \delta_c \simeq 1.686$ will have collapsed to produce dark matter haloes by time t. In this lecture we examine how to assign a halo mass to this structure. But first, we need to introduce some concepts...

[from F. Bosch]

How to Assign (Halo) Mass to Collapsed Regions?

We now return to our main question of interest:

According to SC model, regions in the linear density field with $\delta > \delta_c$ have collapsed to produce virialized dark matter haloes. How can we associate a mass to those haloes, and how can we use the statistics of the linear density field to infer the halo mass function, i.e., the (comoving) number density of haloes as a function of halo mass?

Idea:

Let δ_M be the linear density field smoothed on a mass scale M, i.e., $\delta_M = \delta(\vec{x}; R)$ where $M = \gamma_{\rm f}\, \bar{\rho}\, R^3$, then those locations where $\delta_M = \delta_{\rm c}(t)$ are the locations where, at time t, a halo of mass M condenses out of the evolving density field....

In this case, the halo mass function simply follows from calculating the number density of peaks in the smoothed density field, i.e.,

$$\underbrace{n(>M)}_{\text{number density of haloes with mass} >M} = \underbrace{n_{\rm pk}(\delta_M)}_{\text{number density of peaks above } \delta_c \text{ in density field smoothed on mass scale } M}$$

[from F. Bosch]

Peak Formalism & Cloud-in-Cloud Problem

This idea was explored in a seminal paper by Bardeen et al. (1986), known as "BBKS".

THE STATISTICS OF PEAKS OF GAUSSIAN RANDOM FIELDS

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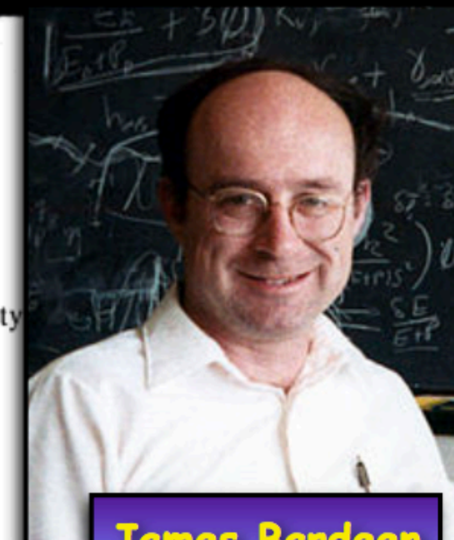
Received 1985 July 25; accepted 1985 October 9


James Bardeen

ABSTRACT

Cosmological density fluctuations are often assumed to be Gaussian random fields. The local maxima of such fields are obvious sites for the formation of nonlinear structures. The statistical properties of the peaks can be used to predict the abundances and clustering properties of objects of various types. In this paper, we derive (1) the number density of peaks of various heights $\nu\sigma_0$ above the rms σ_0; (2) the factor by which the peak density is enhanced in large-scale overdense regions; (3) the n-point peak-peak correlation function in the limit that the peaks are well separated, with special emphasis on the two- and three-point correlations; and (4) the density profiles centered on peaks. To illustrate the predictive power of this semianalytic approach, we apply our formulae to structure formation in the adiabatic and isocurvature $\Omega = 1$ cold dark matter (CDM) models. We assume bright galaxies form only at those peaks in the density field (smoothed on a galactic scale) that are above some global threshold height $\nu_t \approx 3$ fixed by normalizing to the galaxy number density. We find, for example, that the shapes of the peak-peak two- and three-point correlation functions for the adiabatic CDM model agree well with observations before any dynamical evolution, just due to the propensity of the peaks to be clustered in the initial conditions. Only moderate dynamical evolution is required to bring the amplitude of the correlations up to the observed level. The corresponding redshift of galaxy formation z_g in the isocurvature model is too recent ($z_g \approx 0$) for this model to be viable. Even for the adiabatic models $z_g \approx$ 3–4 is predicted. We show that the mass-per-peak ratio in clusters, and thus presumably the cluster mass-to-light ratio, is substantially lower than in the ambient medium, alleviating the Ω problem. We also confirm that the smoothed density profiles of collapsing structures of height $\sim\nu_t$ are inherently triaxial.

[from F. Bosch]

Peak Formalism & Cloud-in-Cloud Problem

This idea was explored in a seminal paper by Bardeen et al. (1996), known as "BBKS".

Using elegant, clever mathematics they were able to compute the number density, clustering properties, shapes and density profiles of peaks in a smoothed Gaussian random field (which itself is also a Gaussian random field), all as function of the peak height $\nu_{\rm pk} = \frac{\delta_{\rm pk}}{\langle \delta_M^2 \rangle^{1/2}} = \frac{\delta_{\rm pk}}{\sigma_M}$ (see MBW §7.1 for details)

Unfortunately, it soon became clear that the identification

peak in δ_M ⟷ halo with mass $> M$

faces a very serious problem:

Consider the same density field, but smoothed on two different mass scales, M_1 and M_2, where $M_2 > M_1$. Let δm be a mass element associated with a peak of $\delta_1 = \delta(\vec{x}; M_1)$ but also with a peak of $\delta_2 = \delta(\vec{x}; M_2)$. Is δm part of a halo of mass M_1 or M_2 ?

- If $\delta_2 < \delta_1$ the obvious interpretation is that δm is part of M_1 at some early time t_1, and part of $M_2 > M_1$ at some later time $t_2 > t_1$.
- If $\delta_2 > \delta_1$ then δm can never be part of a halo with mass M_1; apparently, contrary to the `ansatz', not every peak in δ_1 can be associated with a halo...

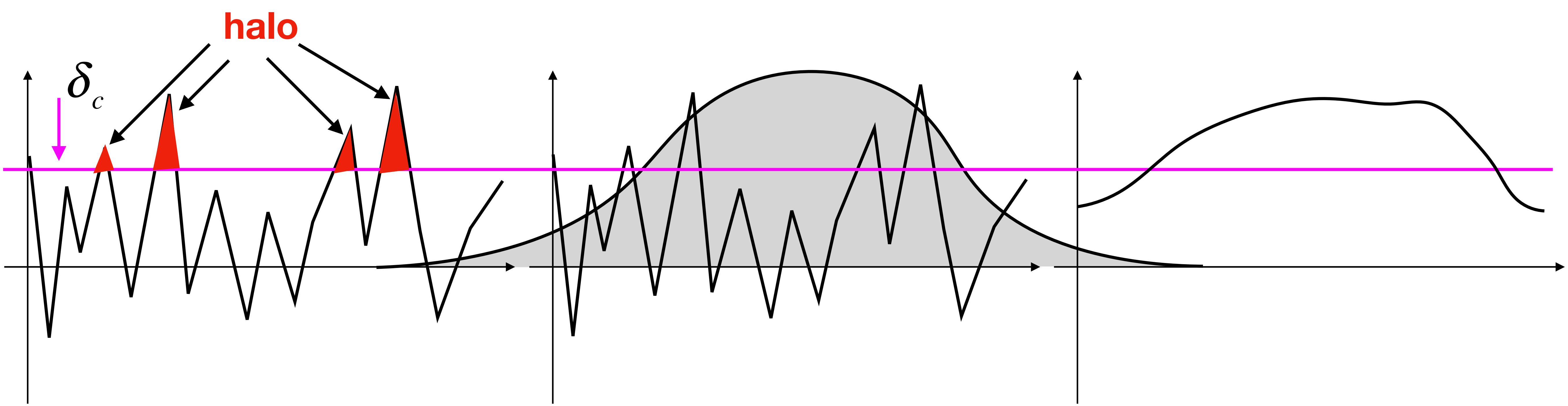
[from F. Bosch]

UP to now, we answered how a single spherical halo form.

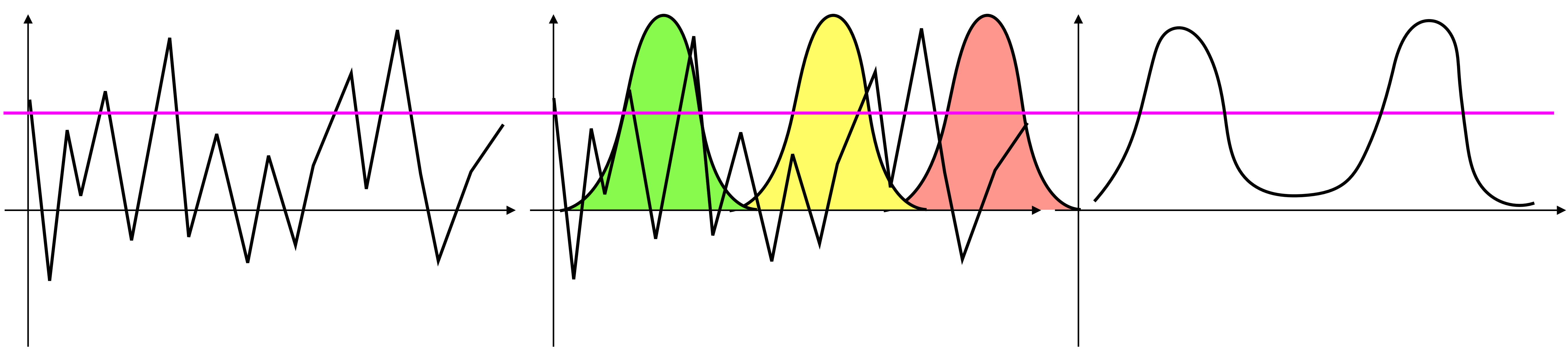
Next question, how the halo distributed in the large scales?

smoothed density field on scale R_W

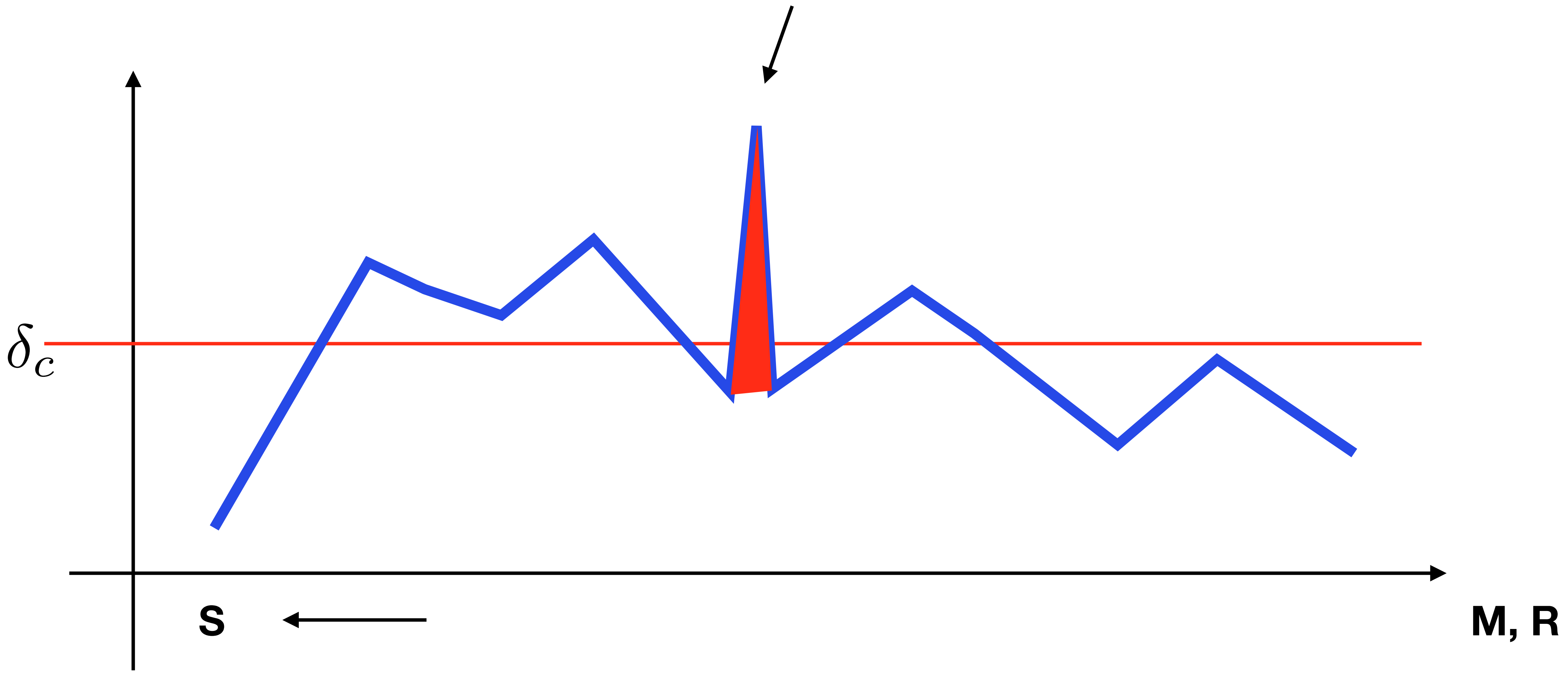
$$\delta(\vec{x}; R_\mathrm{W}) \equiv \int \mathrm{d}^3x' \; W(|\vec{x}' - \vec{x}|; R_\mathrm{W})\delta(\vec{x}')$$



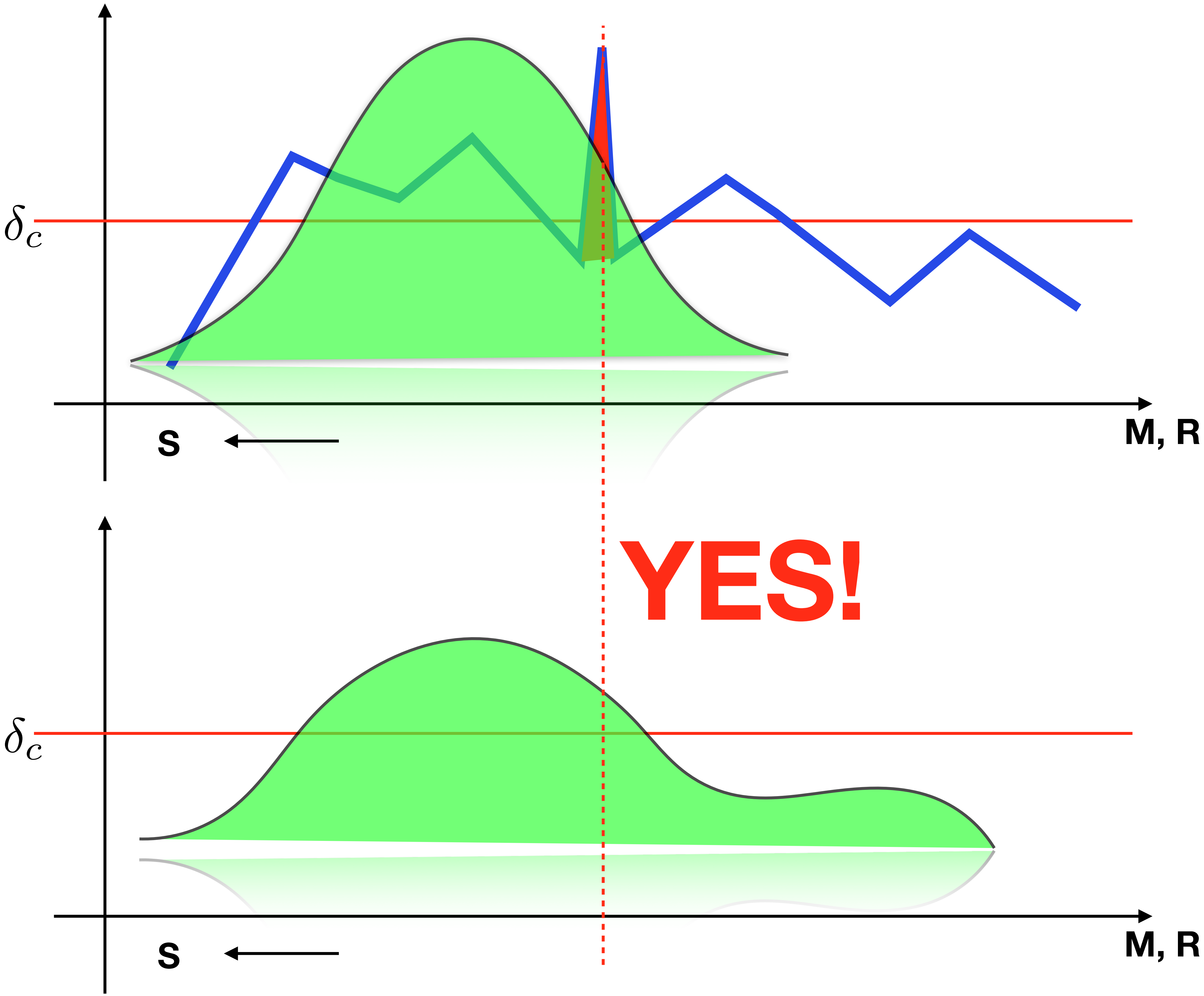
cloud-in-cloud problem



Is this peak a part of A halo?



Smoothing with wider Window function



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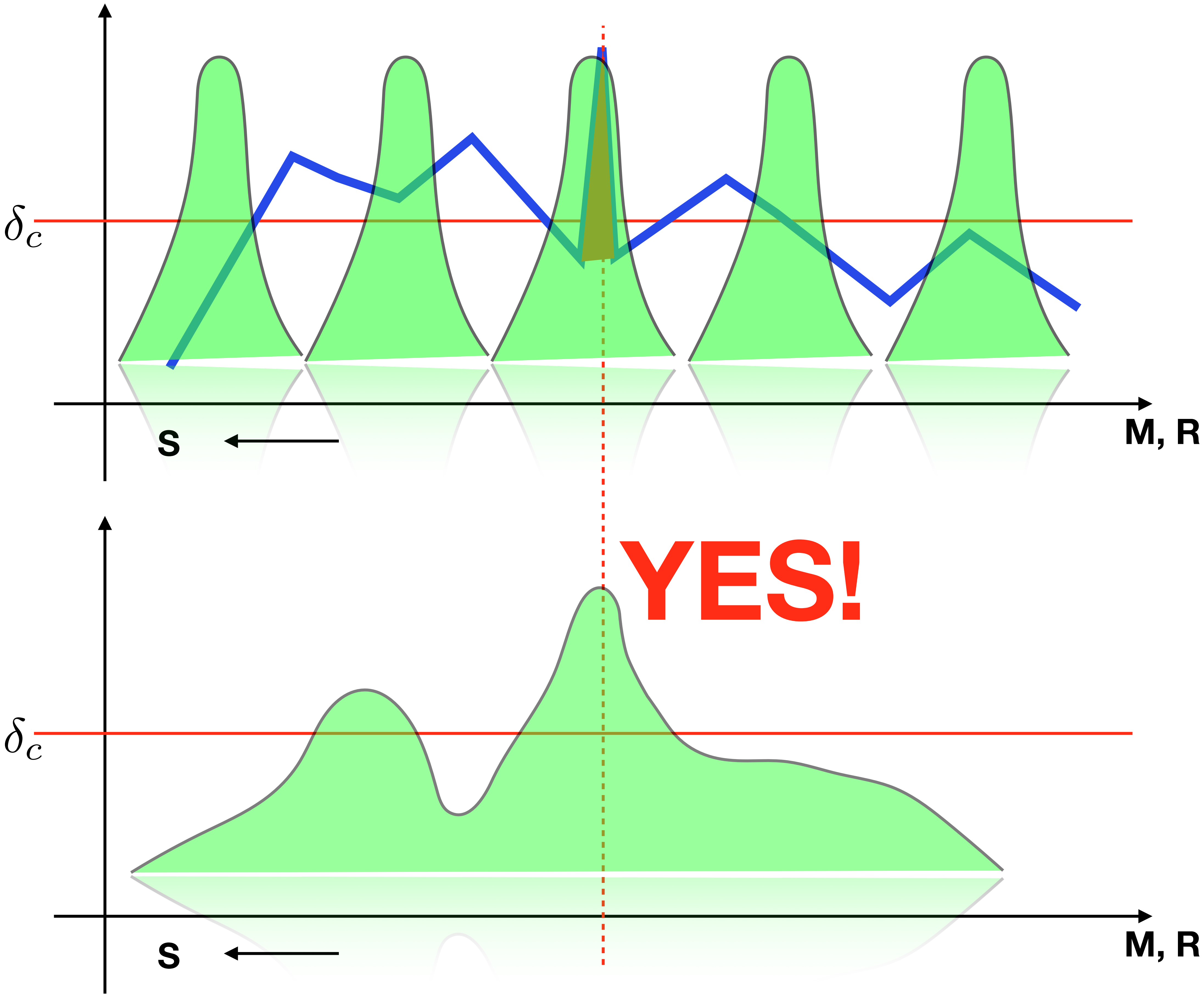

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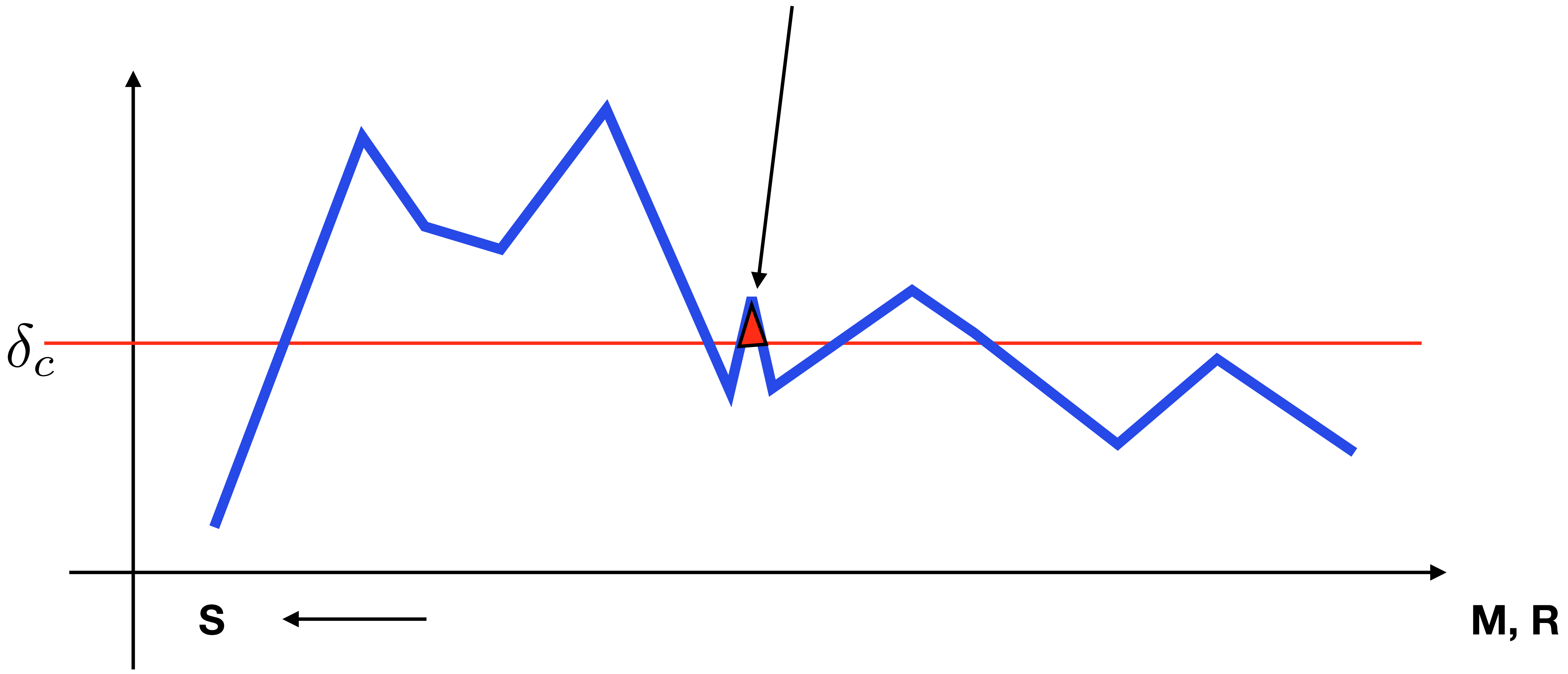
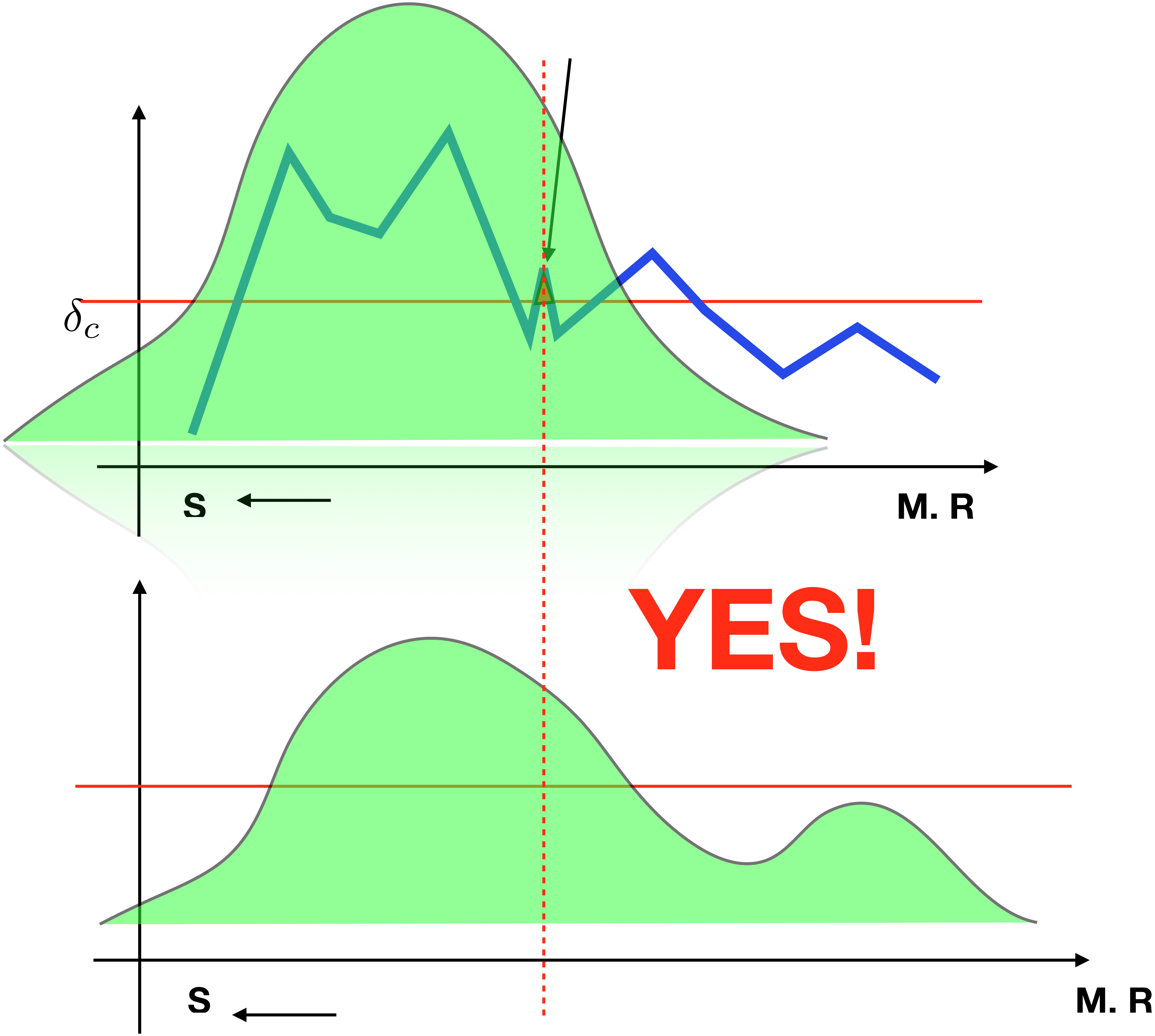
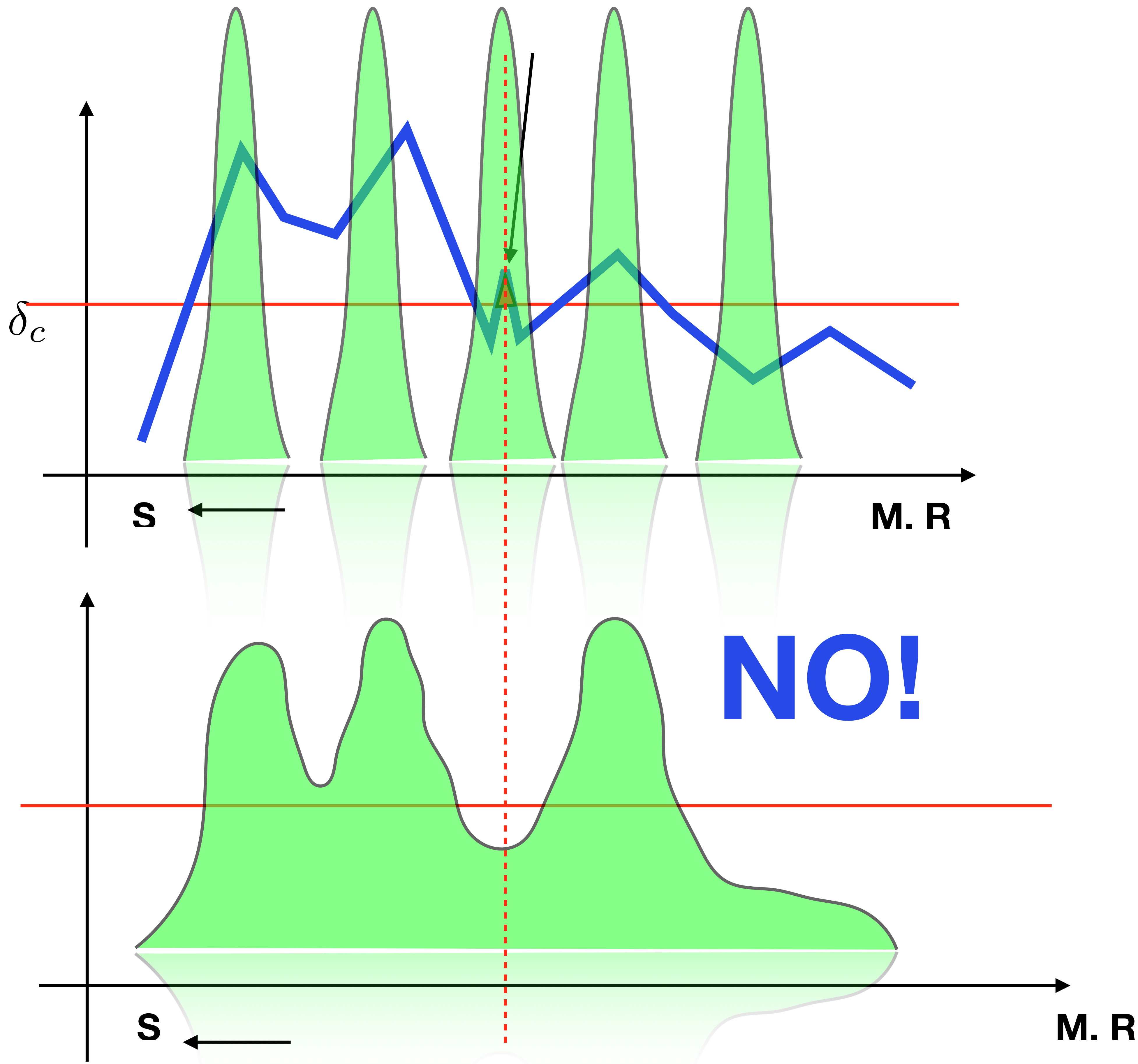
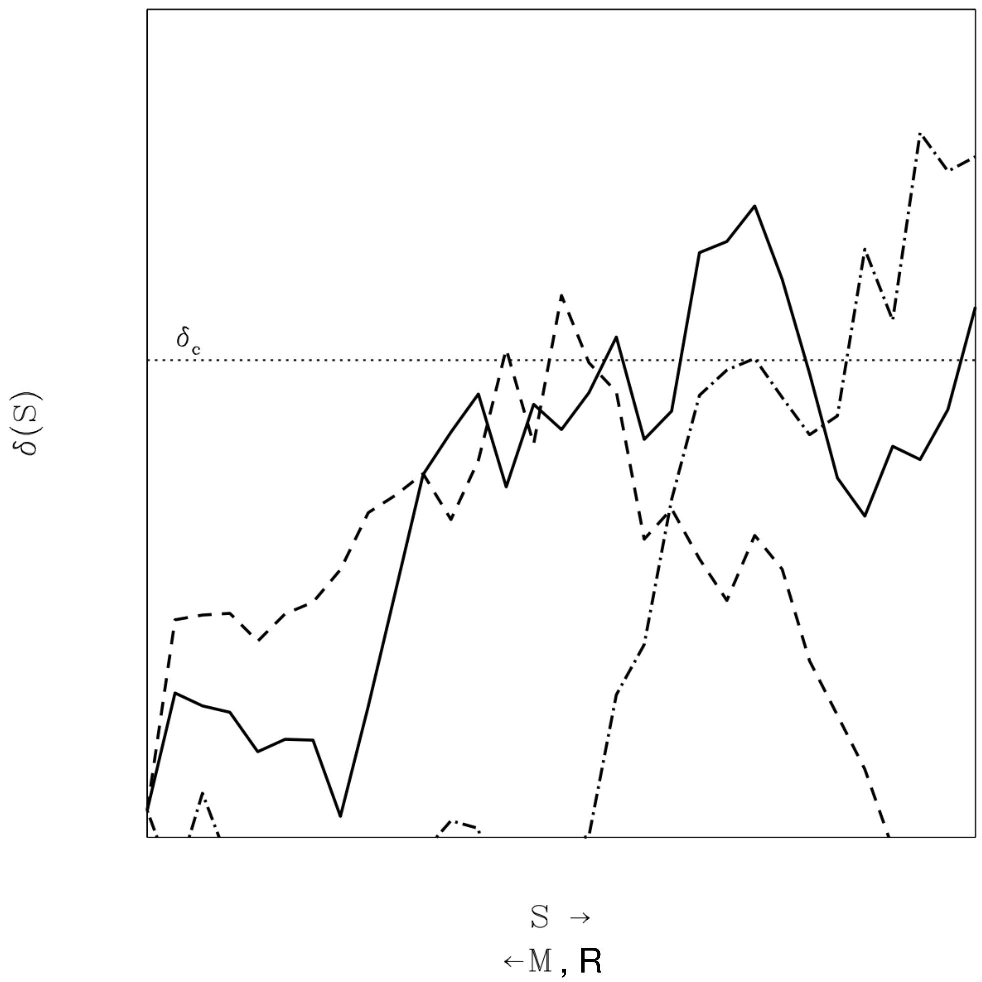


Smoothing with wider Window function



Smoothing with narrower Window function





$$S(R) = \sigma^2(R) = \langle \delta^2(\vec{x}; R) \rangle = \int \mathrm{d}\ln k \; \Delta^2(k) |W(k; R)|^2.$$

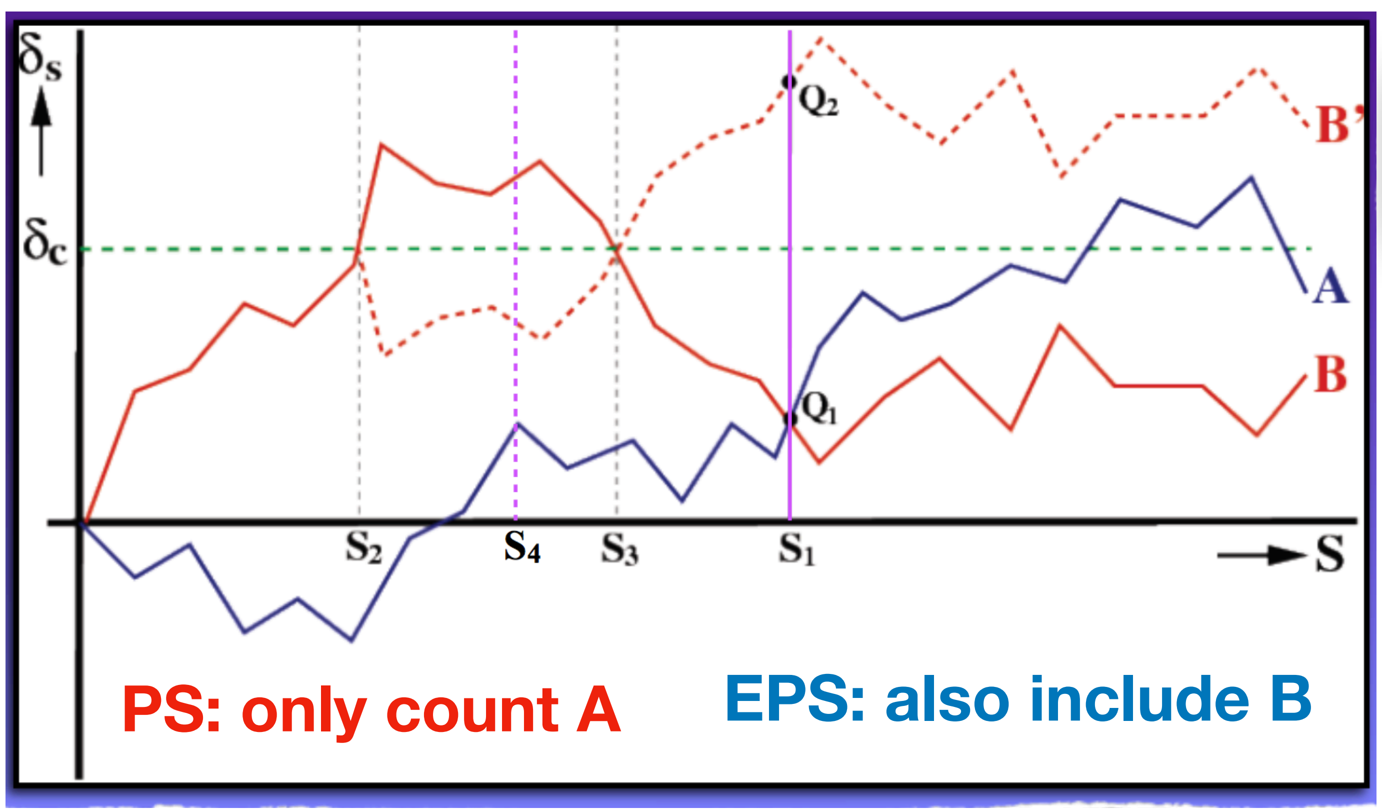
[Press & Schechter 1974]

$$P(\delta; R)\mathrm{d}\delta = \frac{1}{\sqrt{2\pi\sigma^2(R)}} \exp[-\delta^2/2\sigma^2(R)]\mathrm{d}\delta.$$

$$F(M) = \int_{\delta_c}^{\infty} P(\delta; R)\mathrm{d}\delta = \boxed{\frac{1}{2}} \mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \qquad \nu = \delta_c / \sigma(R)$$

Excursion Set approach [Bond, Cole, Efstathiou, Kaiser, 1991]

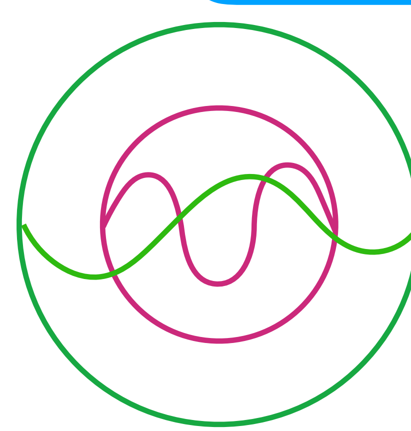
The scaling relation of the filtered density field w.r.t. the smoothing scale, is like a Brownian motion.



$$\frac{\partial \Pi}{\partial S} = \frac{1}{2}\frac{\partial^2 \Pi}{\partial \delta^2}$$

equal probability ↑ ↓ @ every point

trajectory is memoryless!



@linear scale, different k-mode are independent!

All the regions with $\delta_M > \delta_c$, collapsed into a halo

Probability of $\delta_M > \delta_c$ is equal to the fraction of mass in a halo, with the halo mass > M.

The Press-Schechter Mass Function

Because of the cloud-in-cloud problem, the peak formalism of BBKS has largely been abandoned in favor of the less rigorous, but more succesfull, Press-Schechter formalism

Press & Schechter (1974) postulated that:

> "the probability that $\delta_M > \delta_c(t)$ is the same as the mass fraction that at time t is contained in halos with mass greater than M"

For a Gaussian random field, one has that

$$\boxed{\mathcal{P}(\delta_M > \delta_c) = \frac{1}{\sqrt{2\pi}\,\sigma_M}\int_{\delta_c}^{\infty} \exp\left[-\frac{\delta_M^2}{2\sigma_M^2}\right] \mathrm{d}\delta_M = \frac{1}{2}\mathrm{erfc}\left[\frac{\delta_c}{2\sigma_M}\right]}$$

Here $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ is the complimentary error function, and we consider it understood that $\delta_c = \delta_c(t)$. According to the PS postulate, we thus have that

$$F(> M, t) = \frac{1}{2}\,\mathrm{erfc}\left[\frac{\delta_c}{2\,\sigma_M}\right]$$

Note: since $\lim_{M\to 0}\sigma_M = \infty$ and $\mathrm{erfc}(0) = 1$ we see that the PS postulate predicts that only 1/2 of all matter in the Universe is locked-up in collapsed haloes...

[from F. Bosch]

The Press-Schechter Mass Function

We are now ready to write down the PS halo mass function:

We define the mass function as $n(M,t)\,\mathrm{d}M$, which is the number of haloes with masses in the range $[M, M+\mathrm{d}M]$ per (comoving) volume. Hence, $n(M,t) = \frac{\mathrm{d}n}{\mathrm{d}M} = M\,\frac{\mathrm{d}n}{\mathrm{d}\ln M}$.

Beware of units and different notatio

We have that $\frac{\partial F(>M)}{\partial M}\,\mathrm{d}M$ is equal to the fraction of mass that is locked up in haloes with masses in the range $[M, M+\mathrm{d}M]$.

Multiplying by $\bar{\rho}$ yields the total mass per unit volume that is locked up in those haloes.

Hence, the halo mass function is simply given by $n(M,t)\,\mathrm{d}M = \frac{\bar{\rho}}{M}\,\frac{\partial F(>M)}{\partial M}\,\mathrm{d}M$

Using the Press-Schechter ansatz plus fudge factor we thus obtain:

$$\boxed{n(M,t)\,\mathrm{d}M = 2\frac{\bar{\rho}}{M}\,\frac{\partial \mathcal{P}(>\delta_c)}{\partial M}\,\mathrm{d}M = \sqrt{\frac{2}{\pi}}\,\frac{\bar{\rho}}{M^2}\,\frac{\delta_c}{\sigma_M}\exp\left(-\frac{\delta_c^2}{2\sigma_M^2}\right)\left|\frac{\mathrm{d}\ln\sigma_M}{\mathrm{d}\ln M}\right|\,\mathrm{d}M}$$

where we have used that $\partial\mathcal{P}/\partial M = \partial\mathcal{P}/\partial\sigma_M \times |\mathrm{d}\sigma_M/\mathrm{d}M|$.

[from F. Bosch]

The Press-Schechter Mass Function

Upon defining the variable $\nu \equiv \delta_c(t)/\sigma(M)$ the PS mass function can be written in a more compact form:

$$\boxed{n(M,t)\,\mathrm{d}M = \frac{\bar{\rho}}{M^2}\, f_{\rm PS}(\nu) \left|\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln M}\right| \mathrm{d}M \quad \text{where} \quad f_{\rm PS}(\nu) = \sqrt{\frac{2}{\pi}}\,\nu\, e^{-\nu^2/2}}$$

$f_{\rm PS}(\nu)$ is called the multiplicity function and gives the mass fraction associated with haloes in a unit range of $\ln\nu$. Note that time enters only through $\delta_c(t) \simeq 1.686/D(t)$

WARNING: some authors define $\nu = \delta_c^2(t)/\sigma^2(M)$ which results in a somewhat modified multiplicity function.....always check how ν is defined!!

If we define a characteristic mass, M^*, by $\sigma(M^*) = \delta_c(t)$ (i.e., by $\nu(M^*) = 1$) then:

- For $M \ll M^*$ we have that $n(M,t) \propto M^{\alpha-2}$, where $\alpha = \mathrm{d}\ln\sigma/\mathrm{d}\ln M$.
  For a CDM cosmology $\alpha \to 0$ at low mass end so that $n(M) \propto M^{-2}$
- For $M \gg M^*$ the abundance of haloes is exponentially suppressed.
- Since $\delta_c(t)$ decreases with time, the characteristic halo mass grows as function of time; as time passes more and more massive haloes will start to form...

[from F. Bosch]

The Excursion Set Formalism

Bond et al. (1991) came up with an alternative derivation of the halo mass function that does not suffer from a `fudge-factor problem'

Dick Bond

EXCURSION SET MASS FUNCTIONS FOR HIERARCHICAL GAUSSIAN FLUCTUATIONS

J. R. Bond,[1] S. Cole,[2] G. Efstathiou,[3] and N. Kaiser[1]

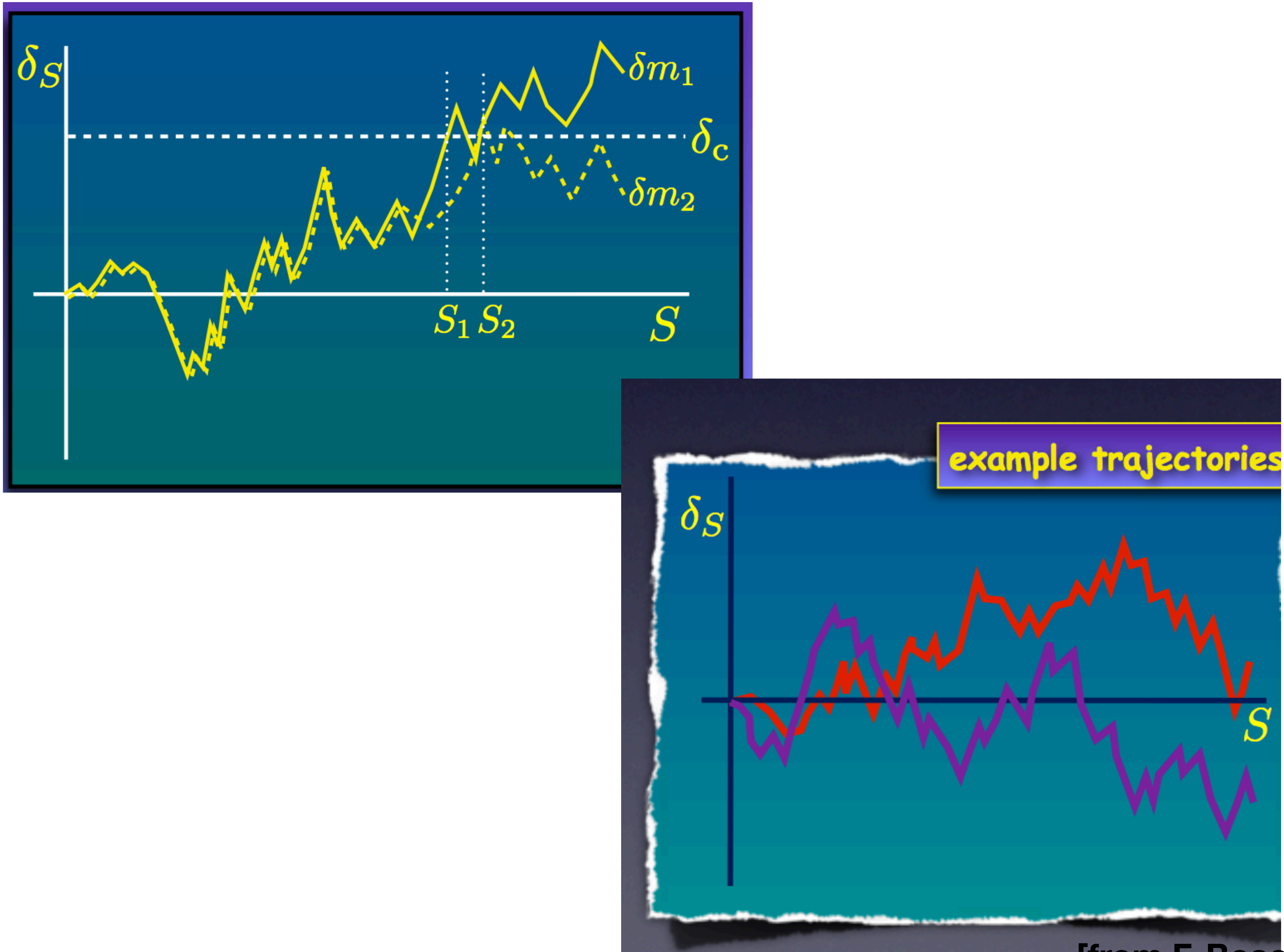
Received 1990 July 23; accepted 1990 December 28

ABSTRACT

Most schemes for determining the mass function of virialized objects from the statistics of the initial density perturbation field suffer from the "cloud-in-cloud" problem of miscounting the number of low-mass clumps, many of which would have been subsumed into larger objects. We propose a solution based on the theory of the excursion sets of $F(\boldsymbol{r}, R_f)$, the four-dimensional initial density perturbation field smoothed with a continuous hierarchy of filters of radii R_f. We identify the mass fraction of matter in virialized objects with mass greater than M with the fraction of space in which the initial density contrast lies above a critical overdensity when smoothed on some filter of radius greater than or equal to $R_f(M)$. The differential mass function is then given by the rate of first upcrossings of the critical overdensity level as one decreases R_f at constant position $\boldsymbol{r}$. The shape of the mass function depends on the choice of filter function. The simplest case is "sharp k-space" filtering, in which the field performs a Brownian random walk as the resolution changes. The first upcrossing rate can be calculated analytically and results in a mass function identical to the formula of Press and Schechter—complete with their normalizing "fudge factor" of 2. For general filters (e.g., Gaussian or "top hat") no analogous analytical result seems possible, though we derive useful analytical upper and lower bounds. For these cases, the mass function can be calculated by generating an ensemble of field trajectories numerically. We compare the results of these calculations with group catalogs found from N-body simulations. Compared to the sharp k-space result, less spatially extended filter functions give fewer large-mass and more small-mass objects. Over the limited mass range probed by the N-body simulations, these differences in the predicted abundances are less than a factor of 2 and span the values found in the simulations. Thus the mass functions for sharp k-space and more general filtering all fit the N-body results reasonably well. None of the filter functions is particularly successful in identifying the particles which form low-mass groups in the N-body simulations, illustrating the limitations of the excursion set approach. We have extended these calculations to compute the evolution of the mass function in regions that are constrained to lie within clusters or underdensities at the present epoch. These predictions agree well with N-body results, although the sharp k-space result is slightly preferred over the Gaussian or top hat results.

Subject headings: cosmology — galaxies: clustering — numerical methods

[from F. Bosch]


δ_S
δm_1
δ_c
δm_2
S_1 S_2
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example trajectories
δ_S
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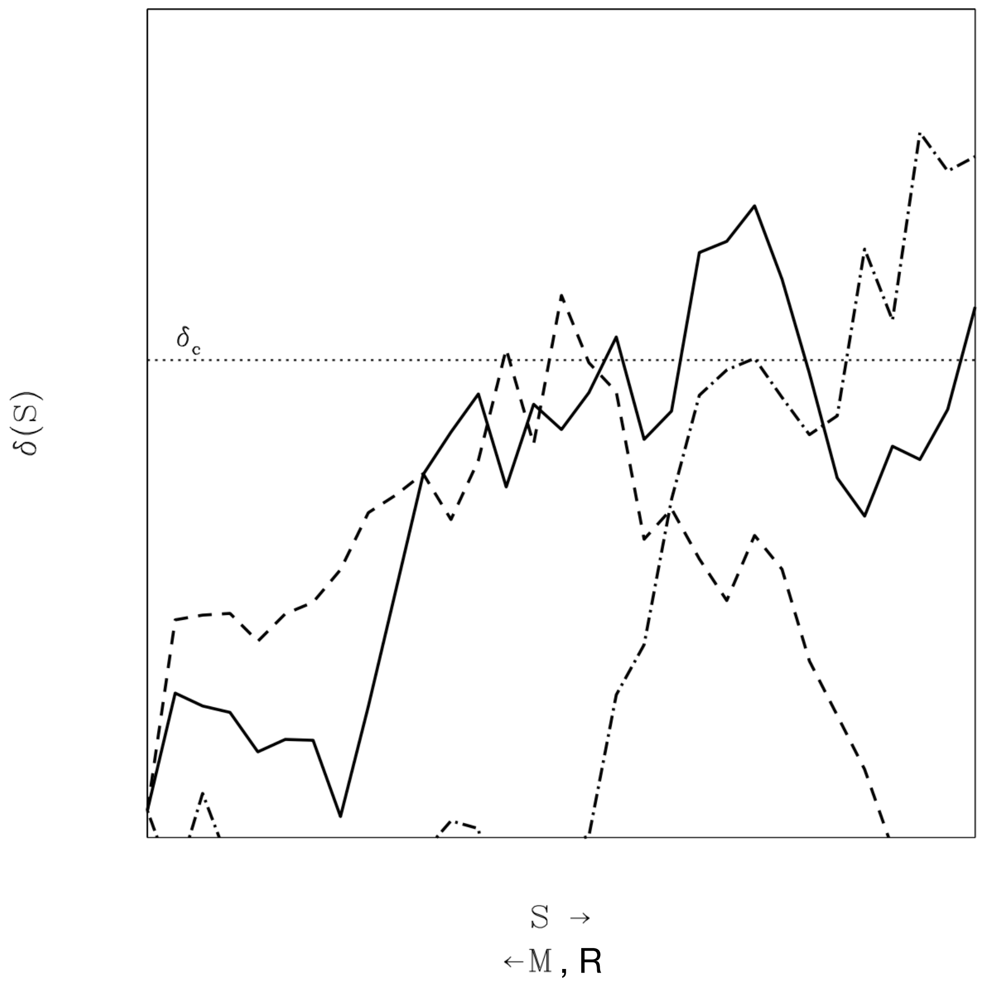
[from F. Bosch]



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[Press & Schechter 1974]

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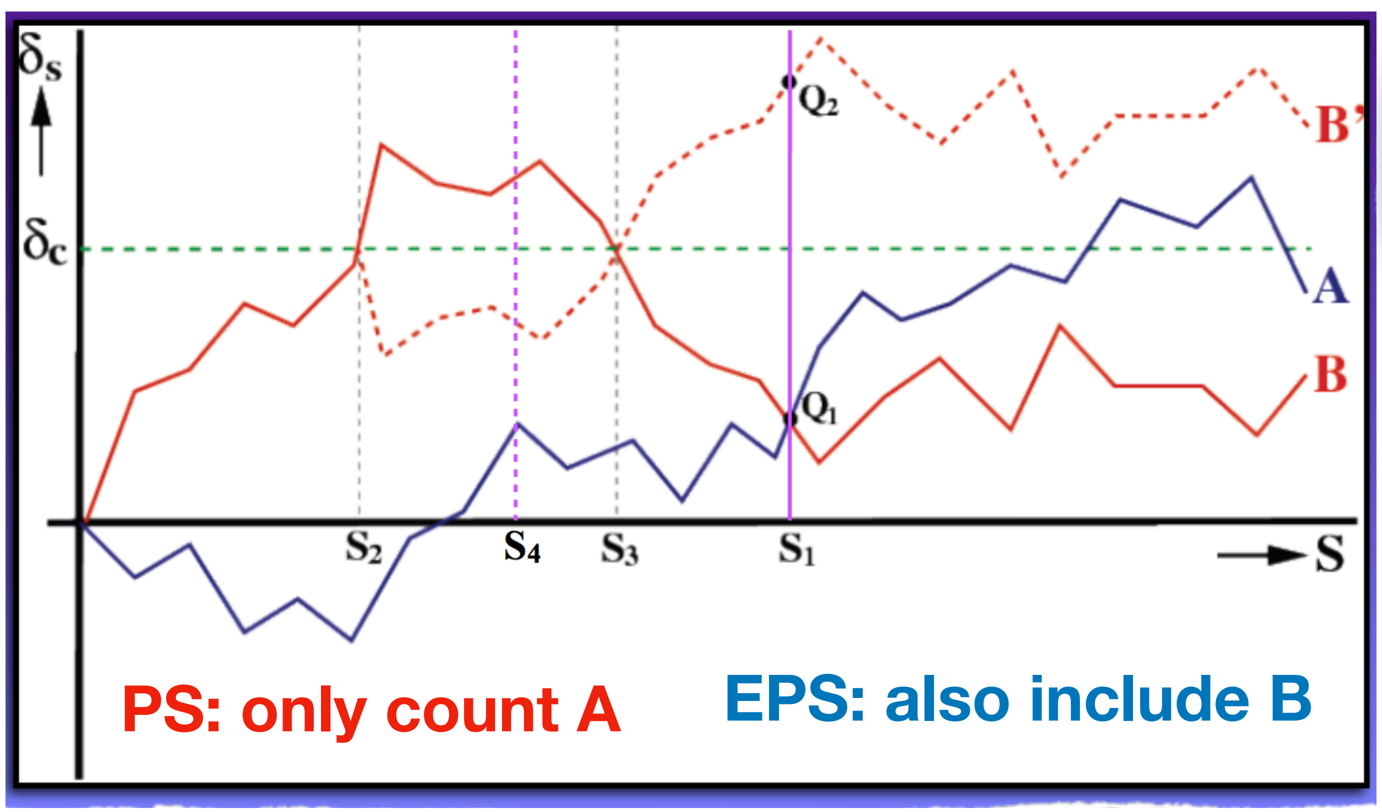

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@ every point

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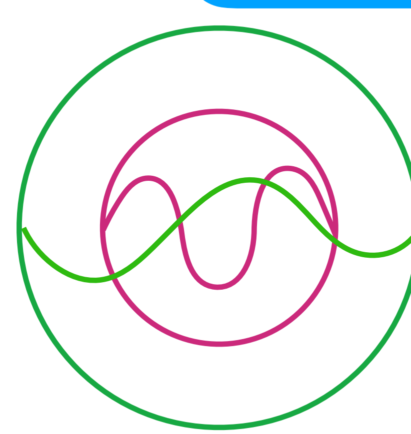
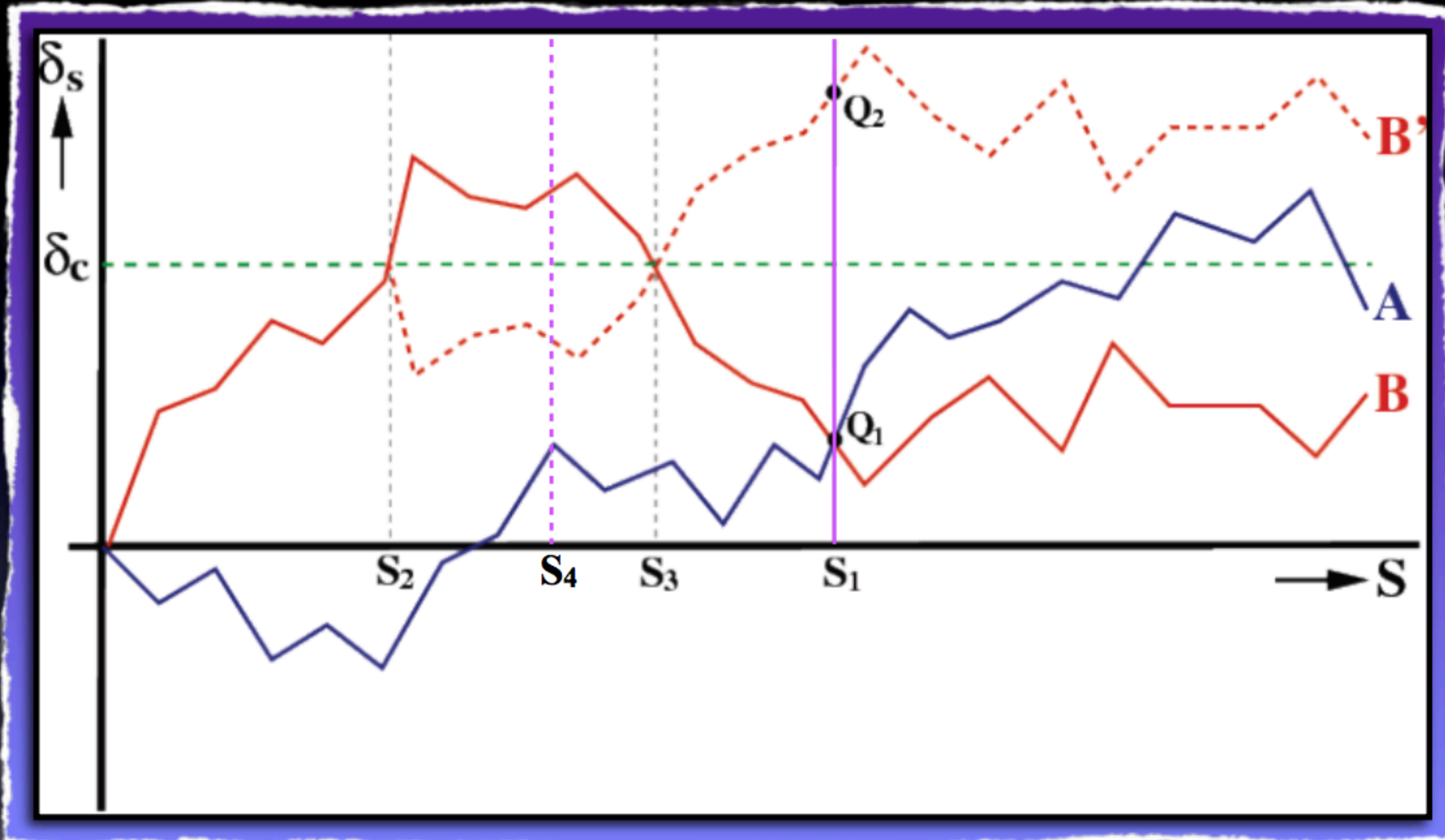


@linear scale, different k-mode are independent!

The Excursion Set Formalism



Three trajectories corresponding to three different mass elements in a Gaussian random field. Note that B' is obtained mirroring trajectory B in the line $\delta_S = \delta_c$ for $S \geq S_2$. Since the trajectories are Markovian B and B' are equally likely!

Consider $\delta_0(\vec{x})$ smoothed on a mass scale M_1 corresponding to $S_1 = \sigma^2(M_1)$

According to PS ansatz, mass elements whose trajectory $\delta_S > \delta_c$ at S_1 reside in dark matter haloes with mass $M > M_1$ ➡ neither A or B are in halo with $M > M_1$

BUT: according to same PS ansatz, mass element associated with trajectory B resides in a halo with $M > M_4 > M_1$: PS ansatz is not self-consistent!!!

[from F. Bosch]

The Excursion Set Formalism

In the excursion set formalism , also called the Extended Press-Schechter (EPS) formalism, one uses the (statistics of) Markovian random walks (the trajectories of mass elements in (S, δ_S)-space) to infer the halo mass function (and more).

PS ansatz: fraction of mass elements with $\delta_S > \delta_c(t)$ is equal to the mass fraction that at time t resides in haloes with masses $> M$, where S and M are related according to $S = \sigma^2(M)$

EPS ansatz: fraction of trajectories with a first upcrossing (FU) of the barrier $\delta_S = \delta_c(t)$ at $S > S_1 = \sigma^2(M_1)$ is equal to the mass fraction that at time t resides in haloes with masses $M < M_1$

Since, each trajectory is guaranteed to upcross the barrier $\delta_S = \delta_c(t)$ at some (arbitrarily large) S, the EPS ansatz predicts that every mass element is in a halo of some (arbitrarily low) mass

$$\Rightarrow \boxed{F(< M_1) = 1 - F(> M_1)}$$

[from F. Bosch]

The EPS Mass Function

Based on the EPS ansatz, we can write the EPS mass function as:

$$
\boxed{
\begin{aligned}
n(M,t)\,\mathrm{d}M &= \frac{\bar{\rho}}{M}\frac{\partial F(>M)}{\partial M}\,\mathrm{d}M = -\frac{\bar{\rho}}{M}\frac{\partial F(<M)}{\partial M}\,\mathrm{d}M \\
&= -\frac{\bar{\rho}}{M}\frac{\partial F_{\mathrm{FU}}(>S)}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}M}\,\mathrm{d}M = \frac{\bar{\rho}}{M} f_{\mathrm{FU}}(S,\delta_{\mathrm{c}})\left|\frac{\mathrm{d}S}{\mathrm{d}M}\right|\mathrm{d}M
\end{aligned}
}
$$

Here $f_{\mathrm{FU}}(S,\delta_{\mathrm{c}})\,\mathrm{d}S$ is the fraction of trajectories that have their first upcrossing of barrier $\delta_{\mathrm{c}}(t)$ between S and $S+\mathrm{d}S$.

Without proof: $$\boxed{f_{\mathrm{FU}}(\nu) = \frac{1}{\sqrt{2\pi}}\frac{\delta_{\mathrm{c}}}{S^{3/2}}\exp\left[-\frac{\delta_{\mathrm{c}}^2}{2S}\right] = \frac{1}{2\,S} f_{\mathrm{PS}}(\nu)}$$ (see MBW §7.2.2 for derivation)

where, as before, we defined $\nu = \delta_{\mathrm{c}}(t)/\sigma(M) = \delta_{\mathrm{c}}/\sqrt{S}$ and we expressed the result in terms of the PS multiplicity function $f_{\mathrm{PS}}(\nu) = \sqrt{2/\pi}\,\nu\exp(-\nu^2/2)$

It is straightforward to show that this yields exactly the same halo mass function as before, but this time there has been no need for a fudge factor....

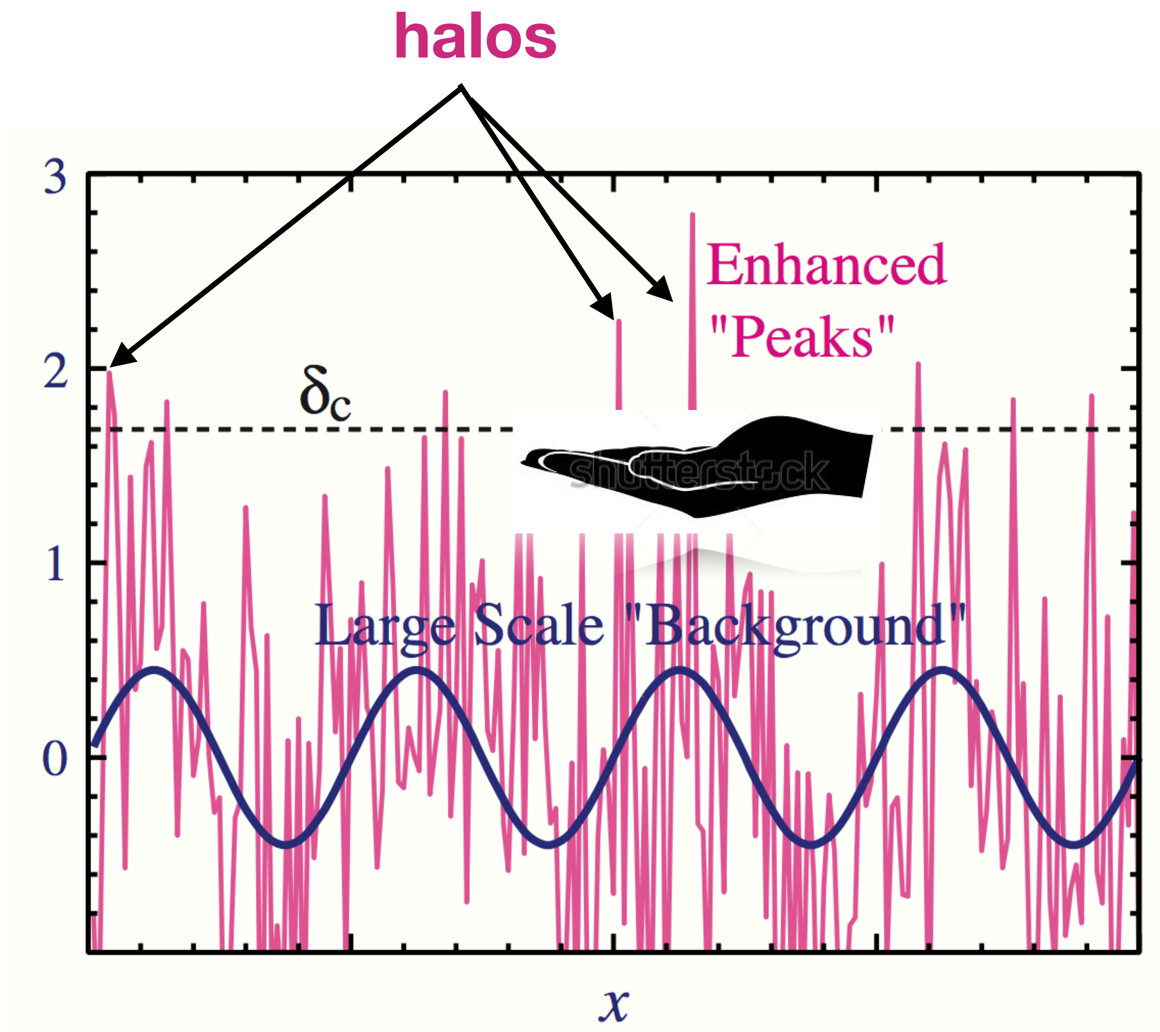
[from F. Bosch]

3. Halo Bias

$n(M)$ **Halo mass function gives the mean number density in the range of** $(M \sim M + dM)$

Linear Halo bias $\delta_m = (1+b) \bullet \delta_h$

Modulation of the LONG wavelength to the local density peak



In denser regime, δ_c **is easier to reach**

$$\delta_c^{\,eff} \sim \delta_c - \delta_0$$

$$n(m,\delta_c^{\,eff}) \sim n(m,\delta_c) - \frac{\partial n}{\partial \delta_c} \bullet \delta_0 + \bullet\bullet\bullet$$

$$b_L \delta_0 = \frac{n(m,\delta_c^{\,eff})}{n(m,\delta_c)} - 1 = \boxed{\frac{\partial \log n}{\partial \delta_c}} \bullet \delta_0$$

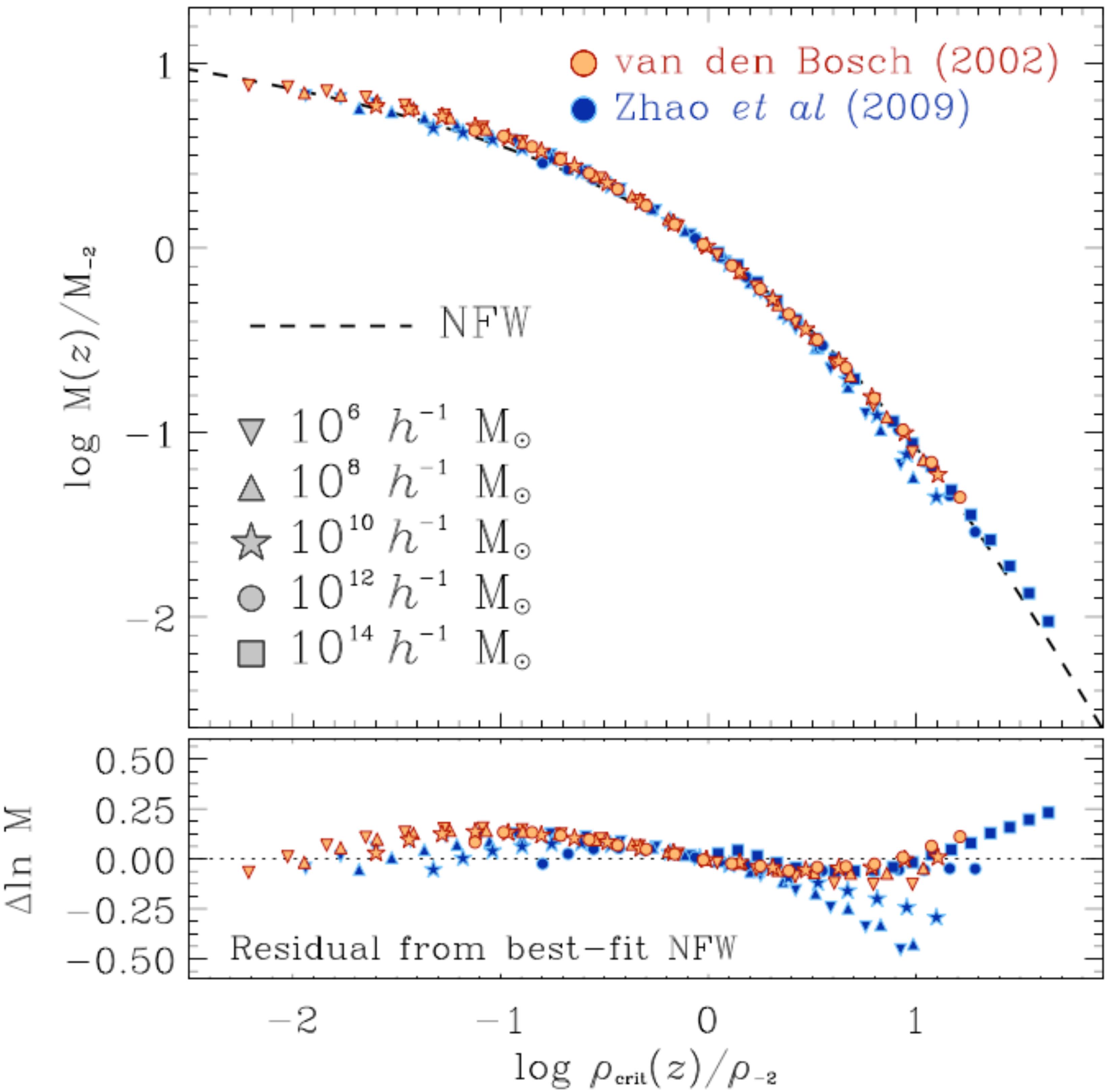
4. Halo concentration

Up to now, we assume top-hat density profile.
Next, we want a more realistic modelling!

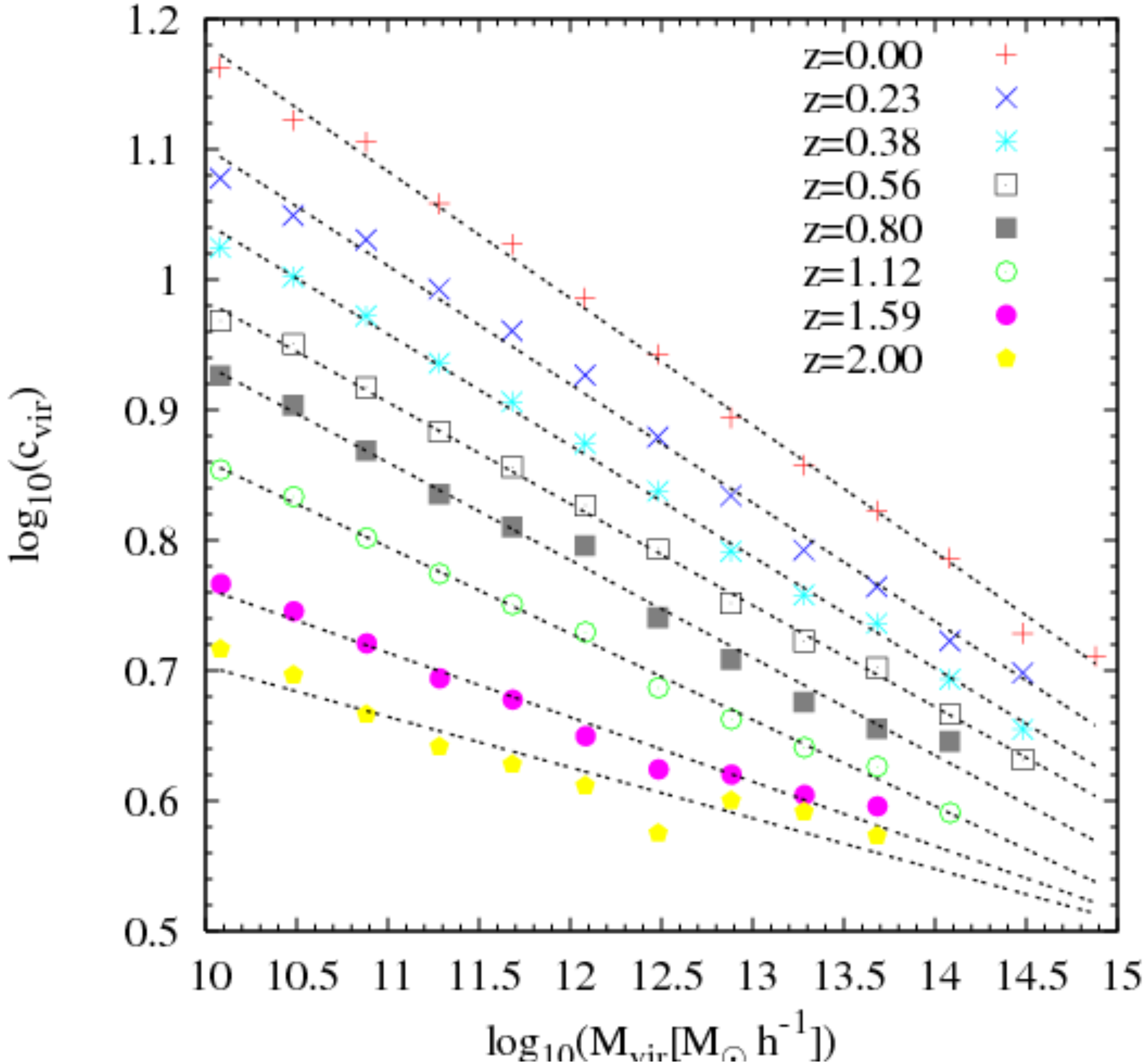
Halo concentration $c_v = \frac{r}{r_s}$

$$\delta\rho_{\rm m}(r) = \frac{\rho_{\rm s}}{\frac{r}{r_{\rm s}}\left(1+\frac{r}{r_{\rm s}}\right)^2}$$

Halo density profile
(NFW profile)

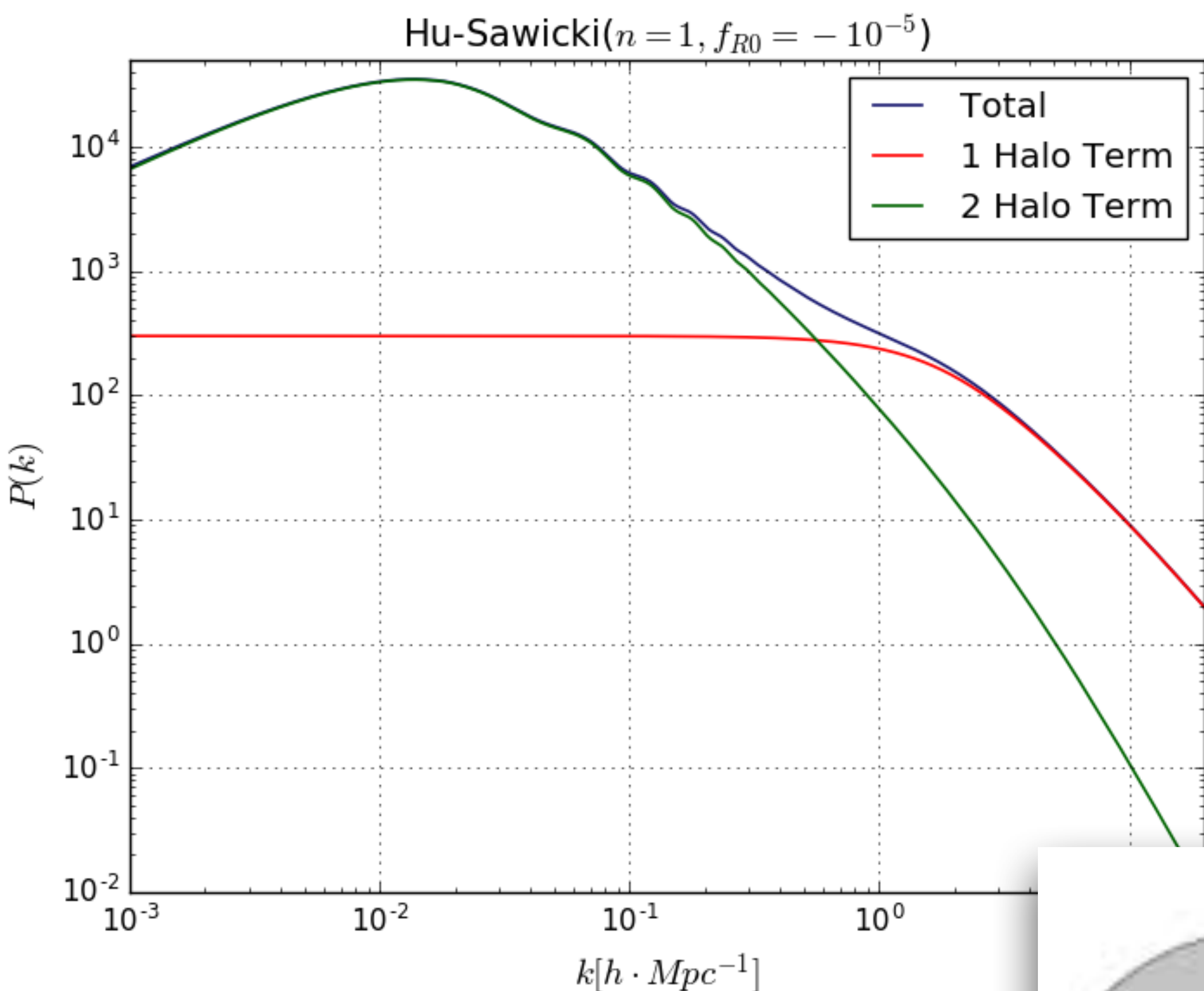


Mass independence:
Large/Small halo looks similar



z=0.00
z=0.23
z=0.38
z=0.56
z=0.80
z=1.12
z=1.59
z=2.00
$\log_{10}(c_{vir})$
$\log_{10}(M_{vir}[M_\odot h^{-1}])$
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
10
10.5
11
11.5
12
12.5
13
13.5
14
14.5
15

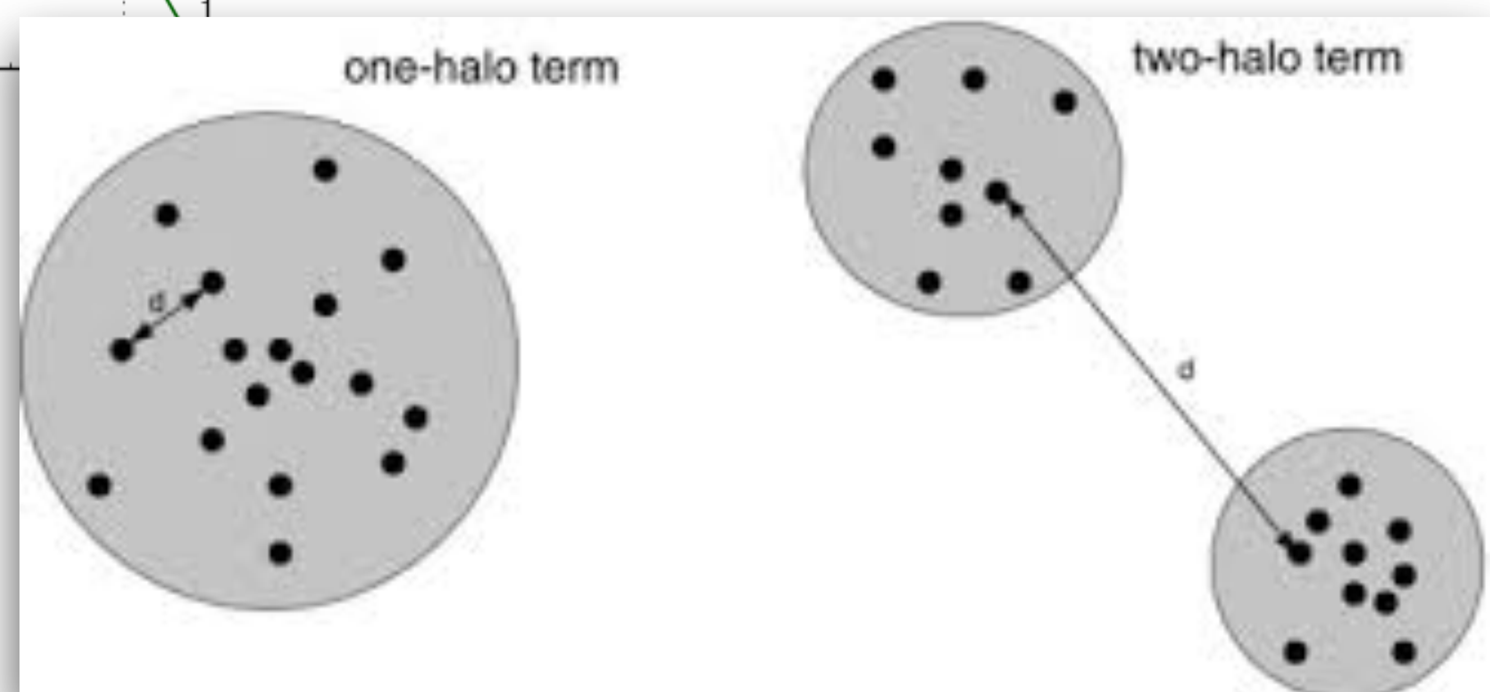
Matter Power spectrum



$$P_{\rm mm}(k) \simeq I^2(k) P_{\rm L}(k) + P^{1h}(k),$$

$$P^{1h}(k) = \int {\rm d}\ln M_{\rm vir} n_{\ln M_{\rm vir}} \frac{M_{\rm vir}^2}{\bar{\rho}_{\rm m}^2} |y(k, M_{\rm vir})|^2$$

$$(k) \simeq \int {\rm d}\ln M_{\rm vir} n_{\ln M_{\rm vir}} \frac{M_{\rm vir}}{\bar{\rho}_{\rm m}} y(k, M_{\rm vir}) b_{\rm L}(M_{\rm vir}),$$



Merger Tree

Beyond a Halo Mass Function... [from F. Bosch]

An important advantage of EPS over PS is that the excursion set formalism provides a neat way to calculate the properties of the progenitors which give rise to a given class of objects (i.e., haloes of a given mass).

For example, one can calculate the mass function at $z=5$ of those haloes (progenitors) which by $z=0$ end up in a massive halo of 10^{15} solar masses.

These progenitor mass functions, in turn, can be used to describe how dark matter haloes assemble over time (in a statistical sense); in particular, they allow the construction of halo merger trees.

These merger trees are invaluable tools in galaxy formation studies...

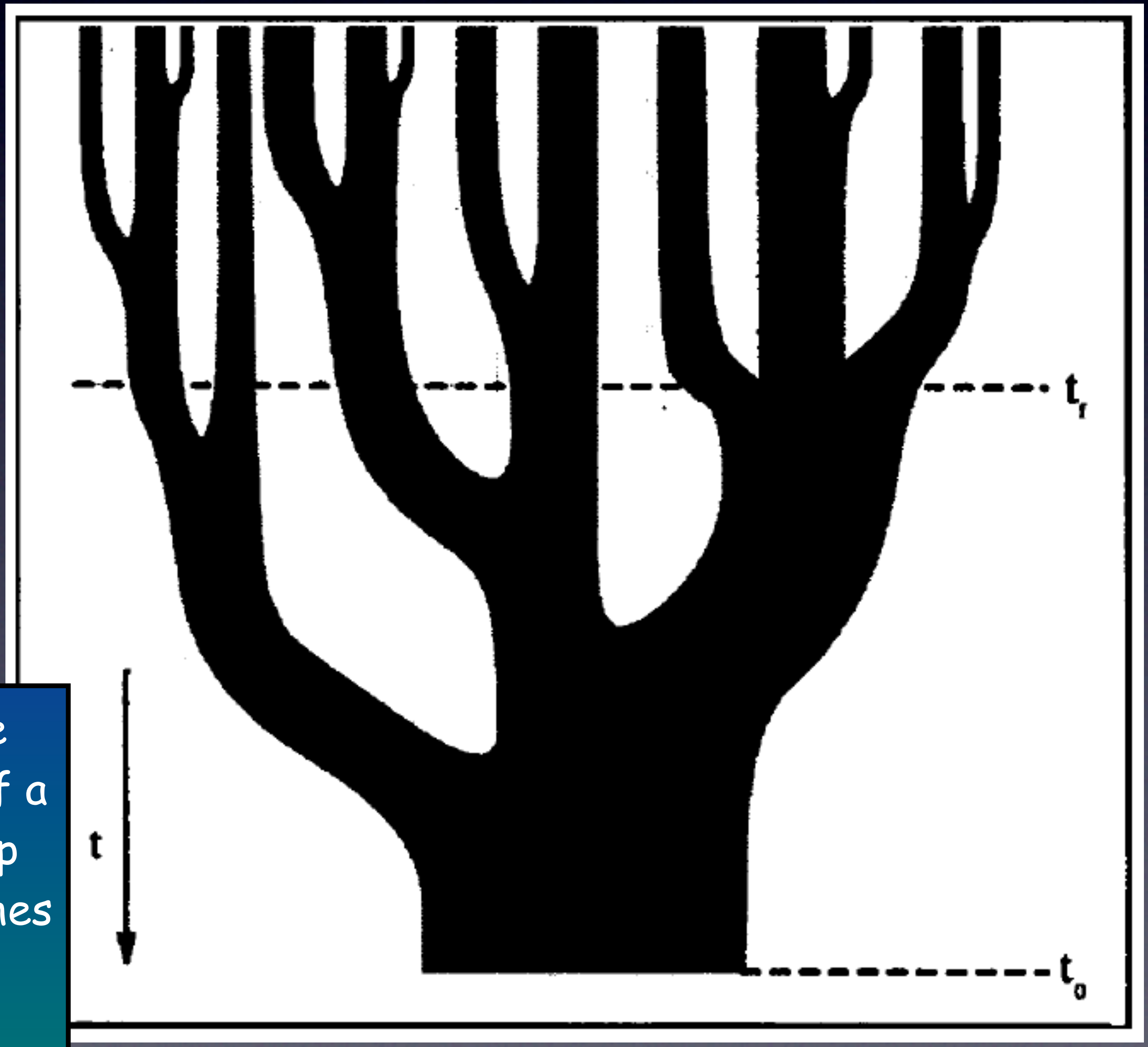


Illustration of a merger tree depicting the growth of a dark matter halo as a result of a series of mergers. Time increases from top to bottom and the width of the tree beaches represents the masses of the individual progenitors...

Source: Lacey & Cole, 1993, MNRAS, 262, 627

Progenitor Mass Function

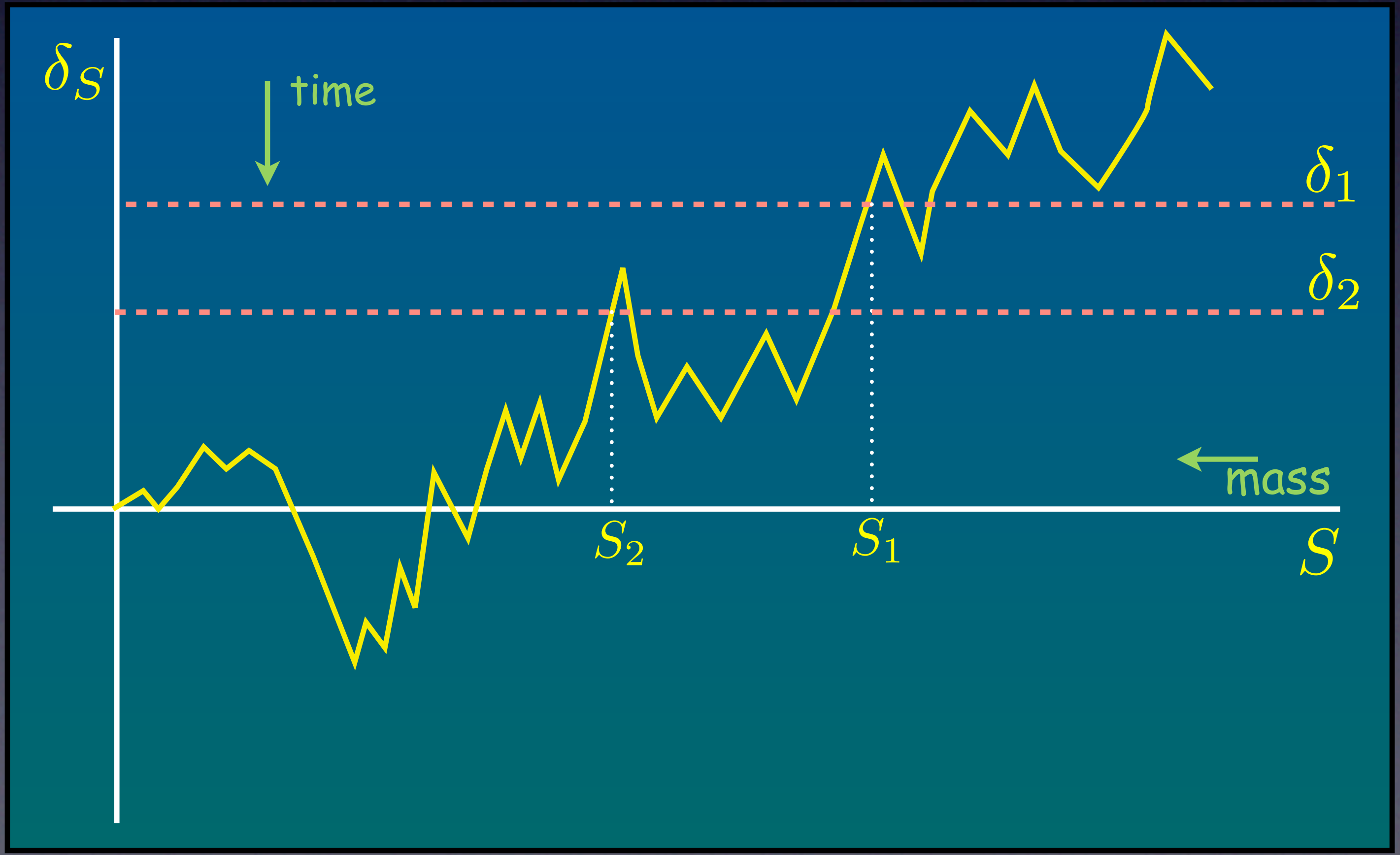
[from F. Bosch]

Consider a spherical region (a patch) of mass M_2 , corresponding to a mass variance $S_2 = \sigma^2(M_2)$ with linear overdensity $\delta_2 \equiv \delta_c(t_2) = \delta_c/D(t_2)$ so that it forms a collapsed object at time t_2.

We are interested in the fraction of M_2 that at some earlier time $t_1 < t_2$ was in a collapsed object of some mass M_1 .

Within the excursion set formalism this means we want to calculate the probability that a trajectory that upcrosses barrier δ_2 at S_2 has its first upcrossing of barrier $\delta_1 = \delta_c(t_1)$ at $S_1 > S_2$ (see illustration).

This is the same problem as before, except for a translation of the origin in the (S, δ_S)-plane .



Progenitor Mass Function

[from F. Bosch]

$$f_{\rm FU}(S,\delta_{\rm c}) = \frac{1}{\sqrt{2\pi}}\,\frac{\delta_{\rm c}}{S^{3/2}}\,\exp\left[-\frac{\delta_{\rm c}^2}{2S}\right]$$

translation

$$f_{\rm FU}(S_1,\delta_1|S_2,\delta_2) = \frac{1}{\sqrt{2\pi}}\,\frac{\delta_1-\delta_2}{(S_1-S_2)^{3/2}}\,\exp\left[-\frac{(\delta_1-\delta_2)^2}{2\,(S_1-S_2)}\right]$$

Converting from mass- to number-weighting

$$n(M_1,t_1|M_2,t_2)\,{\rm d}M_1 = \frac{M_2}{M_1} f_{\rm FU}(S_1,\delta_1|S_2,\delta_2)\left|\frac{{\rm d}S_1}{{\rm d}M_1}\right|{\rm d}M_1$$

$n(M_1,t_1|M_2,t_2)\,{\rm d}M_1$ is the progenitor mass function; it gives the average number of progenitor haloes at time t_1 in the mass range $(M_1, M_1+{\rm d}M_1)$ that at time $t_2 > t_1$ have merged to form a halo of mass M_2.

Merger Trees

[from F. Bosch]

The progenitor mass function allows one to construct halo merger trees using the following algorithm:

For a given host halo mass, M_0 , and a given time step, Δt , draw a set of progenitor masses from the progenitor mass function $n(M_{\rm p}, t_0 + \Delta t | M_0, t_0)$

The progenitors must obey the following two conditions:

- accurately sample the progenitor mass function
- mass conservation: $\sum_i M_{{\rm p},i} = M_0$

For each progenitor, repeat above procedure, thus stepping back in time.

Sounds easy.....is not...

Several different methods have been suggested to contruct halo merger trees; none of them is perfect......