Cosmic Large-scale Structure Formations

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18 weeks

outline

Background (1 w)

- universe geometry and matter components (1 hr)
- Standard candle (SNIa) (0.5 hr)
- Standard ruler (BAO) (0.5 hr)

Linear perturbation (9 w)

- relativistic treatment perturbation (2 hr)
- primordial power spectrum (2 hr)
- linear growth rate (2 hr)
- galaxy 2-pt correlation function (2 hr)
- Bayron Acoustic Osciilation (BAO) (2 hr)
- Redshift Space Distortion (RSD) (2 hr)
- Weak Lensing (2 hr)
- Einstein-Boltzmann codes (2 hr)

Non-linear perturbation (6 w)

- Non-linear power spectrum (2 hr)
- halo model (2 hr)
- N-body simulation algorithms (2 hr)
- Press-Schechter (PS) halo mass function (2 hr)
- Extended-PS (EPS) halo mass function (2 hr)
- halo bias & halo density profile (2 hr)

Statistical analysis (2 w)

- Monte-Carlo Markov Chain sampler (2 hr)
- CosmoMC use (2 hr)

Lecture2

CP:

For a **co-moving observer**, on the **large** scale, the universe is **homogenous** and **isotropic**.

we model the matter distribution via a fluid approach

$\rho(t), P(t), a(t)$ (symmetry simply the system)

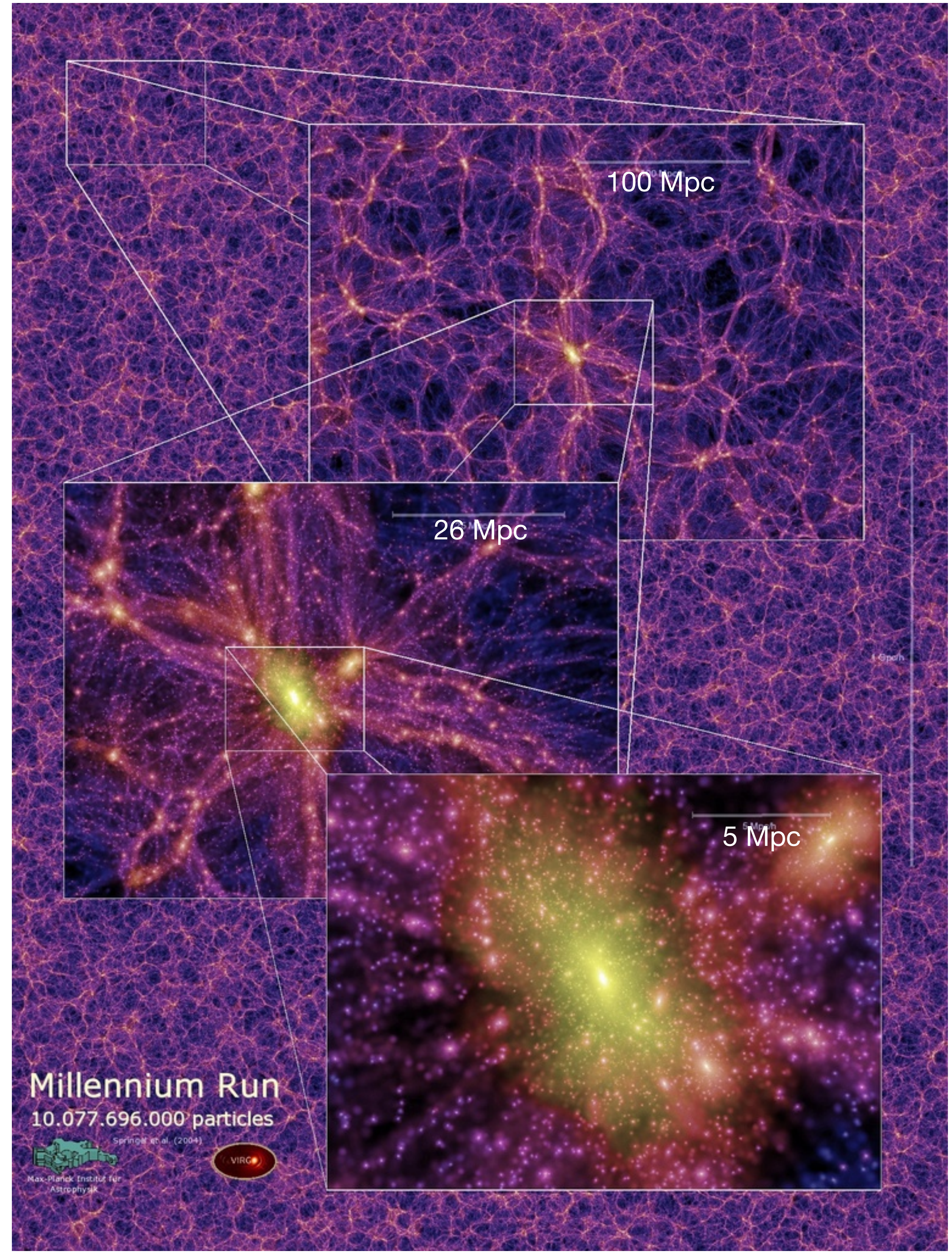
on the **small** scale, the universe is **inhomogeneous** and **anisotropic**

on the perturbation level, we **break** the above symmetry

bad: the system is hard to solve

good: fruitful phenomena

$$
\begin{aligned}
\rho(t,\mathbf{x}) &= \rho_0(t)\big(1+\delta_\epsilon(t,\mathbf{x})\big)\\
\mathbf{v}(t,\mathbf{x}) &= \mathbf{v}_0(t,\mathbf{x}) + \delta\mathbf{v}(t,\mathbf{x})\\
p(t,\mathbf{x}) &= p_0(t) + \delta p(t,\mathbf{x})\,,
\end{aligned}
$$



7. Newton perturbation theory

$\Phi \sim \Psi \sim \delta \sim v \ll 1$

ignore entropy fluct. $\longrightarrow$ $\dot{S} + (\mathbf{v}\cdot\nabla)S = 0$, $p = p(\rho, S)$

$\boxed{\nabla_\mu T^{\mu\nu} = 0}$

$\partial_t \rho = -\boldsymbol{\nabla_r}\cdot(\rho\boldsymbol{u})$ continuity eq.

$(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla_r})\,\boldsymbol{u} = -\frac{\boldsymbol{\nabla_r}P}{\rho} - \boldsymbol{\nabla_r}\Phi$ Euler eq.

$w = P/\rho$ EoS

$\nabla^2\Phi = 4\pi G\delta\rho$ Poisson eq.

$c_s^{\ 2} = \delta P/\delta\rho$ speed of sound

in a static background:

$$\boxed{\delta\ddot{\rho} - c_s^{\ 2}\nabla^2\delta\rho - 4\pi G\bar{\rho}\delta\rho = 0}$$

(derive this eq.)

$\delta\rho$ can not collapse below $c_s^{\ 2}k^2 = 4\pi G\bar{\rho}$

$\delta\ddot{\rho} + c_s^{\ 2}k^2\delta\rho - 4\pi G\bar{\rho}\delta\rho = 0$

Jeans scale

DM: $c_s^{\ 2} = 0$

Linear perturbation theory predict: DM is able to collapse to singularity

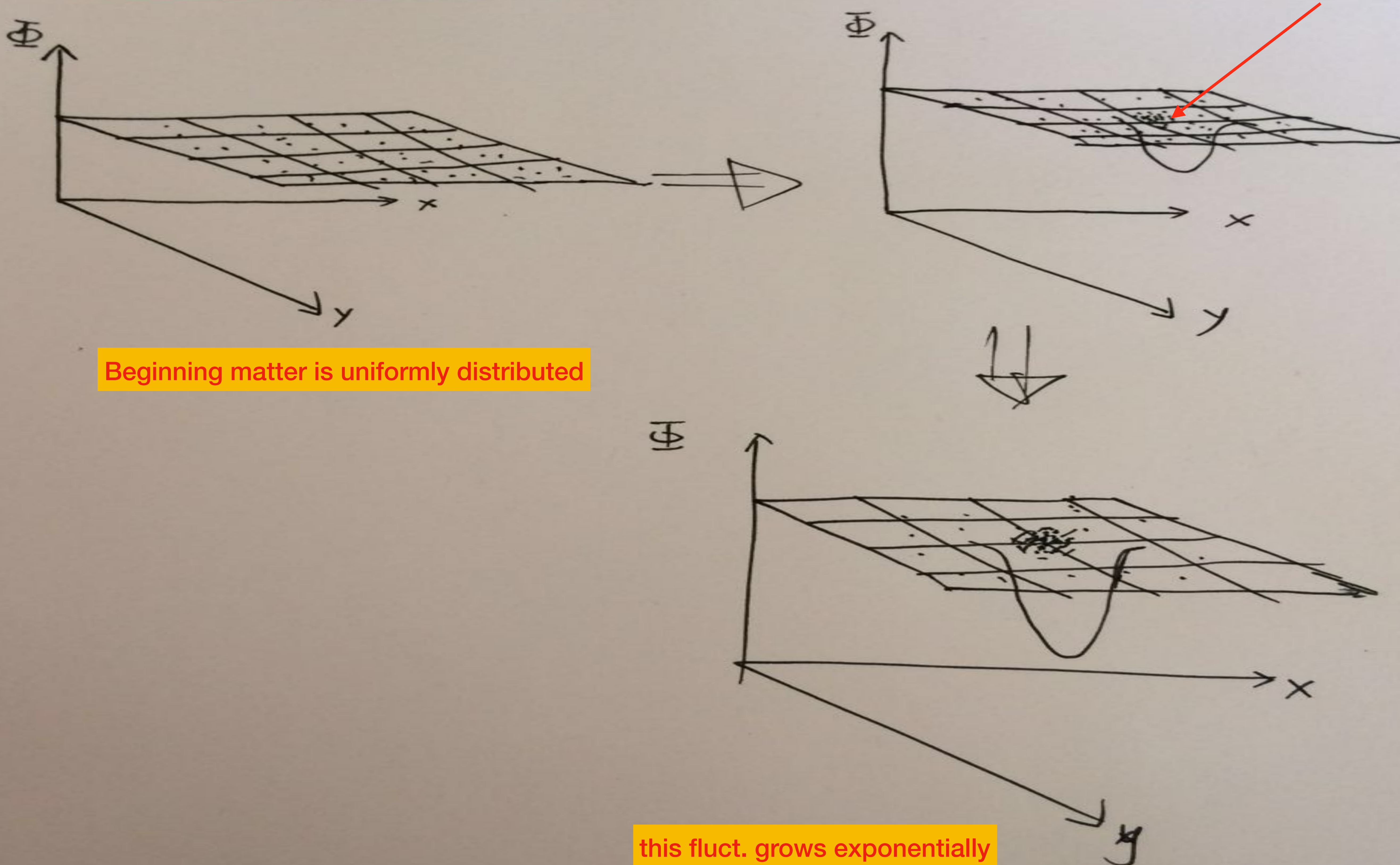
$\ddot{\delta} - 4\pi G\delta = 0$

exponentially grows

$\delta = \delta_0 e^{\sqrt{4\pi G}(t-t_0)}$

this solution is valid until $\delta \to O(1)$

Physical picture



due to unknown reasons, there is a small fluct. @ p

Beginning matter is uniformly distributed

this fluct. grows exponentially

in the expanding background $\boldsymbol{r}(t) = a(t)\boldsymbol{x}$ $\boldsymbol{u}(t) = \dot{\boldsymbol{r}} = H\boldsymbol{r} + \boldsymbol{v}$ (deriv) $\boldsymbol{\nabla}_{\boldsymbol{r}} = a^{-1}\boldsymbol{\nabla}_{\boldsymbol{x}}$ (deriv)

The relationship between time derivatives at fixed $\boldsymbol{r}$ and at fixed $\boldsymbol{x}$ is

$$\left(\frac{\partial}{\partial t}\right)_{\boldsymbol{r}} = \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} = \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial a^{-1}(t)\boldsymbol{r}}{\partial t}\right)_{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}}$$
$$= \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}} - H\boldsymbol{x} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \,. \qquad \text{(deriv)}$$

continuity eq. $\frac{\partial \bar{\rho}}{\partial t} + 3H\bar{\rho} = 0 \longrightarrow$ background $\boxed{\dot{\delta} = -\frac{1}{a}\boldsymbol{\nabla}\cdot\boldsymbol{v}}$ (deriv) $\longrightarrow$ linear

Euler eq. NULL $\longrightarrow$ background **[Pb1.]** $\boxed{\dot{\boldsymbol{v}} + H\boldsymbol{v} = -\frac{1}{a\bar{\rho}}\boldsymbol{\nabla}\delta P - \frac{1}{a}\boldsymbol{\nabla}\delta\Phi} \longrightarrow$ linear

[Pb2.] $\boxed{\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2}\nabla^2\delta = 4\pi G\bar{\rho}\delta}$

background expansion slow down the collapsing process

V.S.

exponential $\longrightarrow$ power-law

$$\boxed{\ddot{\delta} - c_s{}^2\nabla^2\delta - 4\pi G\delta = 0}$$

e.g. MD

$H = 2/(3t)$ $4\pi G\bar{\rho} = 3H^2/2$

[Pb3.]
derive δ_m in RD and dS

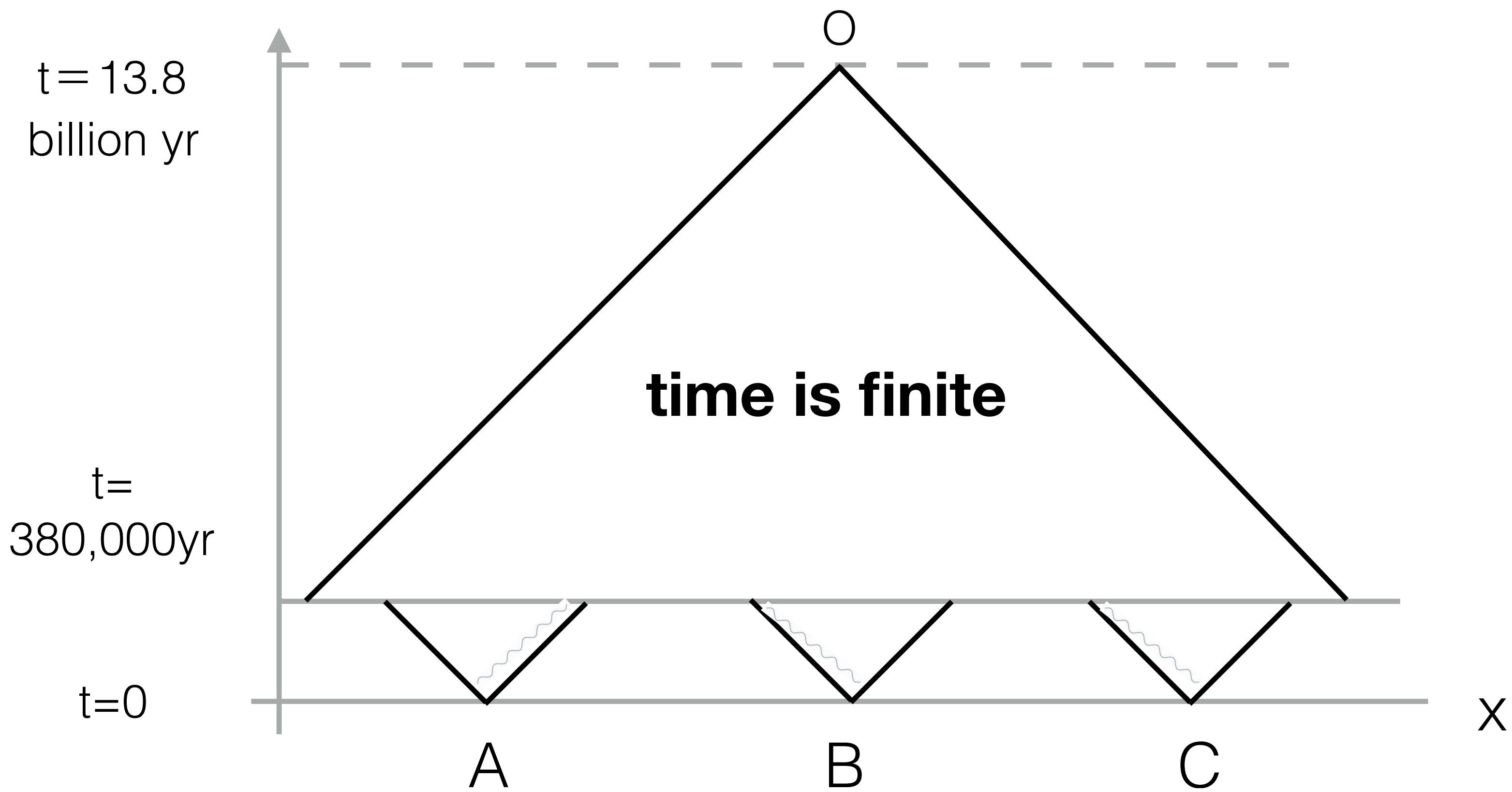
$$\delta_m \propto \begin{cases} t^{-1} \;\propto\; a^{-3/2} \longrightarrow \text{decaying} \\ t^{2/3} \;\boxed{\propto\; a} \longrightarrow \text{growing} \end{cases}$$

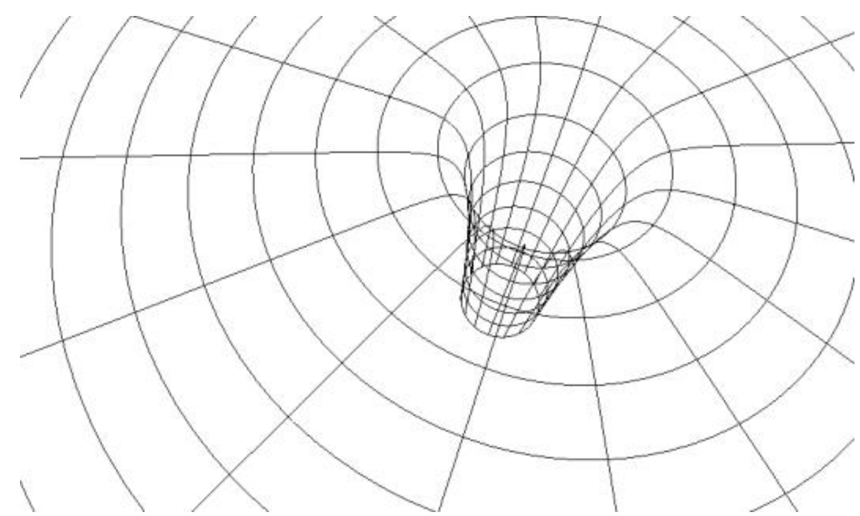
(check)

Newtonian Theory is failed

v/c~1 $\lambda \gg 1/H$ (k<<$\mathcal{H}$)

Hubble horizon (size of the observable patch in the universe)





unlike bh horizon

1. we live inside of the cosmic horizon vs. we live outside of bh horizon

2. finite time vs. infinite time

~~8. Relativistic perturbation theory~~

夏老师已讲

unlike Newtonian theory, GR starts from the metric field

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

$$\mathrm{d}s^2 = a^2(\tau)\Big[\mathrm{d}\tau^2 - \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\Big]$$

$$\mathrm{d}s^2 = a^2(\tau)\Big[(1+2A)\mathrm{d}\tau^2 - 2B_i\mathrm{d}x^i\mathrm{d}\tau - (\delta_{ij}+h_{ij})\mathrm{d}x^i\mathrm{d}x^j\Big]$$

decompose $\delta g_{\mu\nu}$ into scalar, vector, tensor according to SO(3) rotation group

$$\hat{E}^i{}_i = 0 \qquad \partial^i\hat{E}_{ij} = 0$$

(traceless) (transverse)

$$B_i = \underbrace{\partial_i B}_{\text{scalar}} + \underbrace{\hat{B}_i}_{\text{vector}}$$

$$h_{ij} = \underbrace{2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E}_{\text{scalar}} + \underbrace{2\partial_{(i}\hat{E}_{j)}}_{\text{vector}} + \underbrace{2\hat{E}_{ij}}_{\text{tensor}}$$

$$\partial_{\langle i}\partial_{j\rangle}E \equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E$$

$$\partial_{(i}\hat{E}_{j)} \equiv \frac{1}{2}\left(\partial_i\hat{E}_j + \partial_j\hat{E}_i\right) .$$

- *scalars:* $A,\ B,\ C,\ E$
- *vectors:* $\hat{B}_i,\ \hat{E}_i$
- *tensors:* $\hat{E}_{ij}$

$$\delta g_{\mu\nu} = \begin{Bmatrix} 00 & 01 & 02 & 03 \\ & 11 & 12 & 13 \\ & & 22 & 23 \\ & & & 33 \end{Bmatrix}$$

$$d.o.f. = \begin{Bmatrix} s = 4 \\ v = 6-2 = 4 \\ t = 6-1-3 = 2 \end{Bmatrix}$$

$$\nabla^\mu G_{\mu\nu} = 0$$

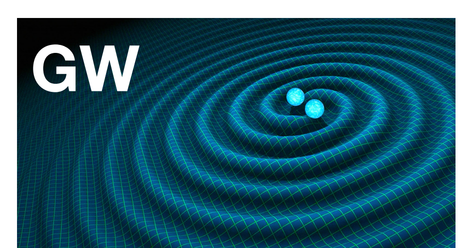
(Bianchi identity)

choosing a coordinate frame

$$\{x^\mu\}$$

$$10 - 4 - 4 = 2$$

pure gravitational d.o.f.



$\hat{E}_{ij}$ tensor pert.

scalar & vector pert. are just the reaction of the gravity to matter sector

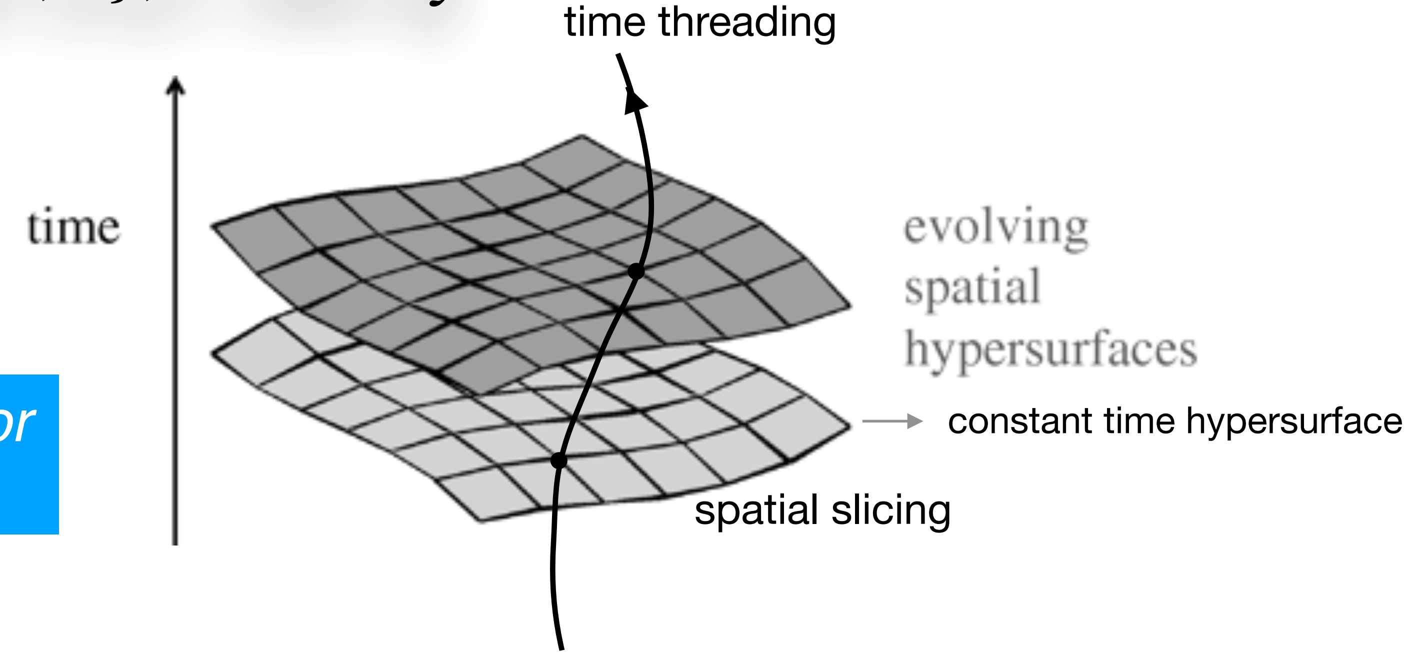
e.g. $\nabla^2\Phi = 4\pi G\delta\rho$

However, tensor pert. can exist even in the matter vacuum!

A pure gravitational phenomena.

gauge fixing $\Longleftrightarrow$ choosing coordinate $x^\mu = \{x^0, x^i\}, x^i = \partial^i \varepsilon + \xi^i$

$$d.o.f. = \left\{ \begin{array}{c} s = 4-2=2 \\ v = 4-2=2 \\ t=2 \end{array} \right\}$$



Theorem: At linear order, scalar/vector/tensor pert. are completely decoupled.

In FRWL background evolution, vector pert. only have decaying mode, so cosmologically irrelevant.

From now on, we only consider scalar & tensor pert. Comparing scalar & tensor pert, signal from scalar > tensor

Mathematically, scalar is more complicated than tensor. Because tensor is gauge invariant, why?

[gauge/coordinate transformation, does not involve tensor. Hence, tensor mode is free of gauge issue]

$$d.o.f. = \left\{ \begin{array}{c} s = 4-2=2 \\ \cancel{v = 4-2=2} \\ \cancel{t=2} \end{array} \right\}$$

scalar mode gauge fixing

2 commonly used gauge

- *Newtonian gauge.*—The choice
$$B = E = 0\ ,$$
gives the metric
$$\mathrm{d}s^2 = a^2(\tau)\left[(1+2\Psi)\mathrm{d}\tau^2 - (1-2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]\ .$$

- synchronous gauge (a frame co-moving with cosmic fluid)

$$ds^2 = a^2[d\tau^2 - (\delta_{ij} + h_{ij})dx^i dx^j] \qquad A = B = 0 \qquad h_{ij} = 2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E$$

gauge transformation Consider the coordinate transformation

$$X^\mu \mapsto \tilde{X}^\mu \equiv X^\mu + \xi^\mu(\tau, \boldsymbol{x}) ,$$

$$\xi^0 \equiv T , \quad \xi^i \equiv L^i = \partial^i L + \hat{L}^i$$

$$\mathrm{d}s^2 = g_{\mu\nu}(X)\mathrm{d}X^\mu \mathrm{d}X^\nu = \tilde{g}_{\alpha\beta}(\tilde{X})\mathrm{d}\tilde{X}^\alpha \mathrm{d}\tilde{X}^\beta$$

$$g_{\mu\nu}(X) = \frac{\partial \tilde{X}^\alpha}{\partial X^\mu}\frac{\partial \tilde{X}^\beta}{\partial X^\nu}\tilde{g}_{\alpha\beta}(\tilde{X})$$

[Pb4.]

ref: Baumann lecture eq. (4.2.48)~(4.2.60)

$$\begin{aligned} A &\mapsto A - T' - \mathcal{H}T , \\ B &\mapsto B + T - L' , \\ C &\mapsto C - \mathcal{H}T - \frac{1}{3}\nabla^2 L , \\ E &\mapsto E - L , \end{aligned}$$

$$\hat{B}_i \mapsto \hat{B}_i - \hat{L}_i' , \qquad \hat{E}_i \mapsto \hat{E}_i - \hat{L}_i ,$$

gauge-inv

$$\hat{E}_{ij} \mapsto \hat{E}_{ij}$$

gauge-inv variables

(Bardeen potential)

$$\Psi \equiv A + \mathcal{H}(B - E') + (B - E')'$$

$$\Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E .$$ (check)

A & C in conformal Newtonian gauge, equals Ψ & Φ ,respectively.

$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu + \delta T^\mu{}_\nu \qquad \bar{T}^\mu{}_\nu = (\bar{\rho} + \bar{P})\bar{U}^\mu \bar{U}_\nu - \bar{P}\delta^\mu_\nu$$

$\bar{U}_\mu = a\delta^0_\mu$, $\bar{U}^\mu = a^{-1}\delta^\mu_0$ for a comoving observer

perfect fluid: no energy dissipation, can not conduct heat

perfect fluid does not exist in real life, but compare with honey, water can be treated as perfect fluid.

$$g_{\mu\nu}U^\mu U^\nu = 1$$

$$\delta T^\mu{}_\nu = (\delta\rho + \delta P)\bar{U}^\mu \bar{U}_\nu + (\bar{\rho} + \bar{P})(\delta U^\mu \bar{U}_\nu + \bar{U}^\mu \delta U_\nu) - \delta P \delta^\mu_\nu - \boxed{\Pi^\mu{}_\nu}$$ shear-viscosity

$$U^\mu = a^{-1}[1 - A, v^i]$$ (deriv)

$$T^{\mu\nu}_{vf} = \rho\, u^\mu\, u^\nu \; + \; (p + \boxed{p_b})\, \Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$p_b = -\zeta\, \nabla_\mu\, u^\mu$$ bulk-viscosity

$P(\rho)$,$P_b(\nabla u)$ P: describe the ability to do external work, the mount of work only depends on the initial & final config

P_b: internal energy loss, the mount of energy loss depends also on the volume changing velocity

[Pb5.]

$$\delta T^0{}_0 = \delta\rho \ ,$$
$$\delta T^i{}_0 = (\bar\rho + \bar P)v^i \ ,$$
$$\delta T^0{}_j = -(\bar\rho + \bar P)(v_j + B_j)$$
$$\delta T^i{}_j = -\delta P \delta^i_j - \Pi^i{}_j \ .$$

$$T^\mu{}_\nu(X) = \frac{\partial X^\mu}{\partial \tilde X^\alpha}\frac{\partial \tilde X^\beta}{\partial X^\nu}\tilde T^\alpha{}_\beta(\tilde X)$$

ref: Baumann lecture eq. (4.2.68)~(4.2.73)

[Pb6.]

$$\delta\rho \mapsto \delta\rho - T\bar\rho' \ ,$$
$$\delta P \mapsto \delta P - T\bar P' \ ,$$
$$q_i \mapsto q_i + (\bar\rho + \bar P)L_i'$$
$$v_i \mapsto v_i + L_i' \ ,$$
$$\Pi_{ij} \mapsto \Pi_{ij} \ .$$

pert. classification

- adiabatic pert. $\longrightarrow$ time delay

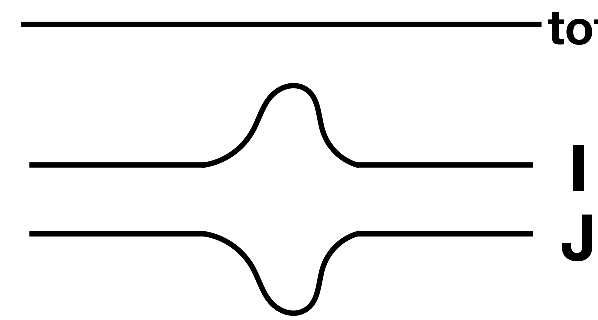
$$\delta\rho_I(\tau, \boldsymbol{x}) \equiv \bar\rho_I(\tau + \delta\tau(\boldsymbol{x})) - \bar\rho_I(\tau) = \bar\rho_I'\delta\tau(\boldsymbol{x})$$

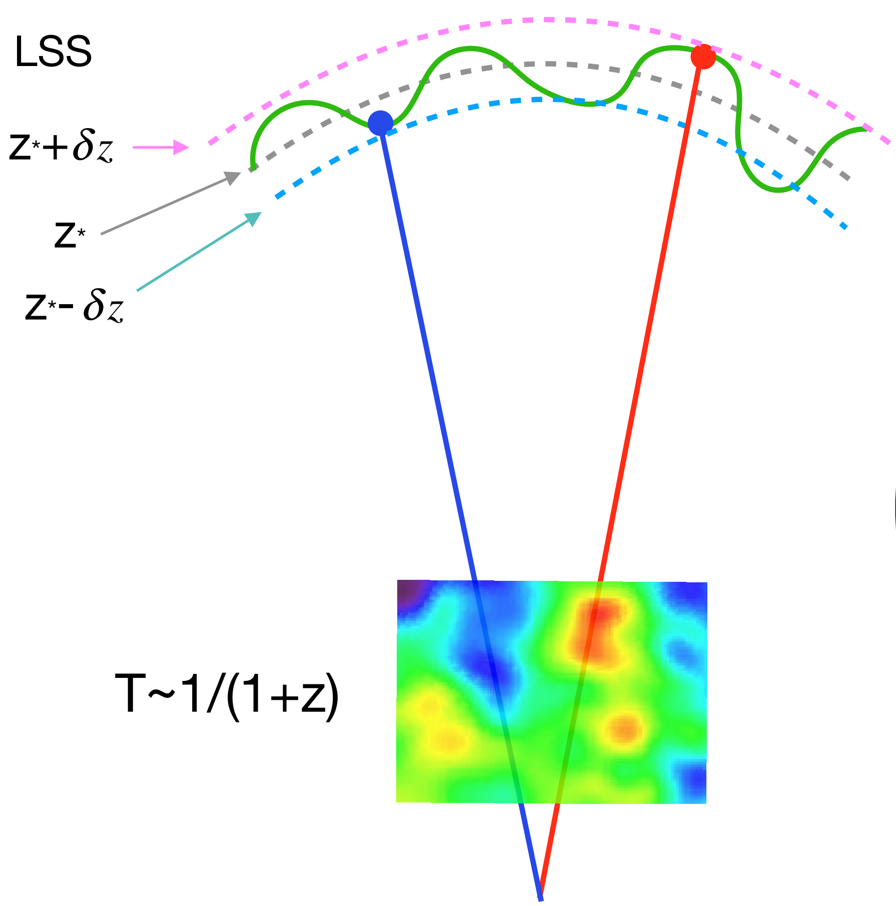
$$\delta\tau = \frac{\delta\rho_I}{\bar\rho_I'} = \frac{\delta\rho_J}{\bar\rho_J'} \quad \text{for all species } I \text{ and } J$$

(not independent!)

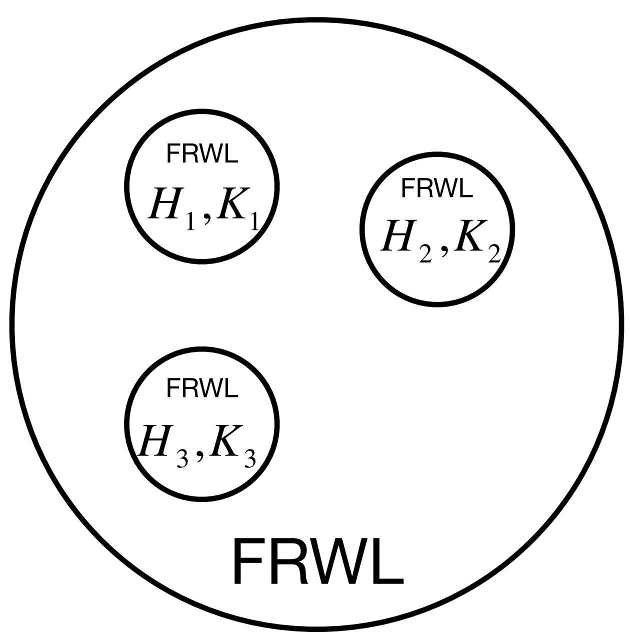
- isocurvature/entropy pert.

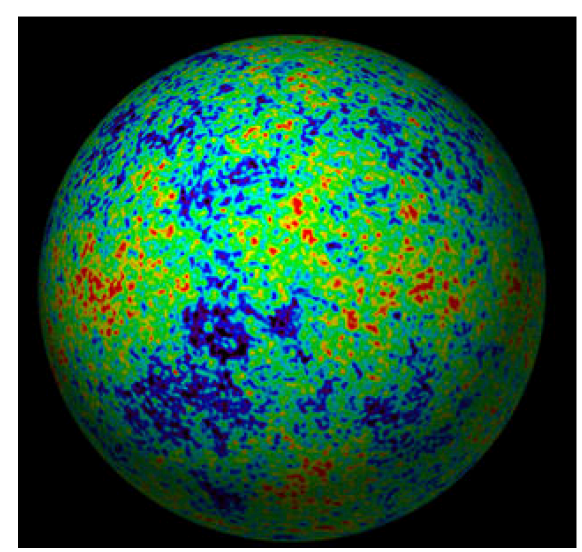
$$\delta P(\rho, s) = \frac{\partial P}{\partial \rho}\delta\rho + \boxed{\frac{\partial P}{\partial s}\delta s}$$





separate universe assumption





Linearised Einstein eq.

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho}\right)$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}$$

$$R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho}$$

[Pb7.]

$$\Gamma^0_{00} = \mathcal{H} + \Psi' \,, \quad \text{(deriv)}$$
$$\Gamma^0_{0i} = \partial_i \Psi \,,$$
$$\Gamma^i_{00} = \delta^{ij} \partial_j \Psi \,,$$
$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} - \left[\Phi' + 2\mathcal{H}(\Phi + \Psi)\right] \delta_{ij}$$
$$\Gamma^i_{j0} = \mathcal{H}\delta^i_j - \Phi' \delta^i_j \,,$$
$$\Gamma^i_{jk} = -2\delta^i_{(j}\partial_{k)}\Phi + \delta_{jk}\delta^{il}\partial_l \Phi \,. \quad \text{(deriv)}$$

[Pb8.]

$$R_{00} = -3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi' + \Psi') + 3\Phi'' \,, \text{(deriv)}$$
$$R_{0i} = 2\partial_i\Phi' + 2\mathcal{H}\partial_i\Psi \,,$$
$$R_{ij} = \left[\mathcal{H}' + 2\mathcal{H}^2 - \Phi'' + \nabla^2\Phi - 2(\mathcal{H}' + 2\mathcal{H}^2)(\Phi + \Psi) - \mathcal{H}\Psi' - 5\mathcal{H}\Phi'\right] \delta_{ij} + \partial_i\partial_j(\Phi - \Psi) \,.$$

$$a^2 R = -6(\mathcal{H}' + \mathcal{H}^2) + 2\nabla^2\Psi - 4\nabla^2\Phi + 12(\mathcal{H}' + \mathcal{H}^2)\Psi + 6\Phi'' + 6\mathcal{H}(\Psi' + 3\Phi') \quad \text{(deriv)}$$

[Pb9.]

$$G_{00} = 3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi'$$

$$G_{0i} = 2\partial_i(\Phi' + \mathcal{H}\Psi)$$

$$G_{ij} = -(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} + \left[\nabla^2(\Psi - \Phi) + 2\Phi'' + 2(2\mathcal{H}' + \mathcal{H}^2)(\Phi + \Psi) + 2\mathcal{H}\Psi' + 4\mathcal{H}\Phi'\right] \delta_{ij} + \partial_i\partial_j(\Phi - \Psi) \,. \tag{4.2.134}$$

- *Newtonian gauge.*—The choice

$$B = E = 0 \,,$$

gives the metric

$$ds^2 = a^2(\tau)\left[(1 + 2\Psi)d\tau^2 - (1 - 2\Phi)\delta_{ij}dx^i dx^j\right] \,.$$

$$ds^2 = a^2(\tau)\left[(1+2\Psi)d\tau^2 - (1-2\Phi)\delta_{ij}dx^i dx^j\right] \ . \tag{4.4.168}$$

In these lectures, we won't encounter situations where anisotropic stress plays a significant role, so we will always be able to set $\Psi = \Phi$.

- The Einstein equations then are

[Pb10.]

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2\,\delta\rho \ , \quad \text{(deriv)} \tag{4.4.169}$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2(\bar{\rho} + \bar{P})v \ , \tag{4.4.170}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2\,\delta P \ . \tag{4.4.171}$$

neglect time evolution term $\quad\Big\downarrow\quad$ $\Psi = \Phi$

Poisson eq. $\quad \nabla^2\Psi = 4\pi G\delta\rho$

$k < \mathcal{H}$ / $\mathcal{L} > 1/\mathcal{H}$

On the cosmic large scale, we do need relativity theory!

On the small scale, Newtonian theory works well!

$k > \mathcal{H}$ / $\mathcal{L} < 1/\mathcal{H}$

$\mathcal{H}^{-1} \longrightarrow$ **co-moving Hubble radius**

Further reading:

- **Baumann lecture note/chapter 4**
- **https://arxiv.org/abs/hep-th/0306071v1**