电动力学习题课

第三章　静磁场

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第三章作业

- 教材 P79 例题 1； 第三章习题 1, 4-7, 9, 10, 13.

3.4（无穷长线电流）

- 直角坐标系 (x, y, z) 和柱坐标系 (r, ϕ, z); $(x < 0)$: 介质，$\boldsymbol{B}_1$; $(x > 0)$: 真空，$\boldsymbol{B}_2$
- 对于以原点为圆心、半径为 r 的圆形环路，根据安培环路定理：

$$\int_{L_1} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{\ell} + \int_{L_2} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{\ell} = I$$

- 边界 $(x = 0)$ 条件：$B_{1x} = B_{2x}, \quad \hat{\boldsymbol{x}} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0$

$$\boldsymbol{B}_1 = \boldsymbol{B}_2 = \frac{\mu\mu_0}{\mu + \mu_0} \frac{I}{\pi r} \hat{\boldsymbol{\phi}}$$

$$\boldsymbol{H}_1 = \frac{1}{\mu} \boldsymbol{B}_1, \quad \boldsymbol{B}_2 = \frac{1}{\mu_0} \boldsymbol{H}_2$$

- $r = 0$: $\boldsymbol{H}_1, \boldsymbol{H}_2 \to \infty; \quad r \to \infty : \boldsymbol{H}_1, \boldsymbol{H}_2 \to 0$
- 磁化强度和磁化电流：$\boldsymbol{M}_1 = (\mu/\mu_0 - 1)\boldsymbol{H}_1, \boldsymbol{M}_2 = 0$, 磁化线电流（在电流 I 与介质分界面处）：

$$I_M = \oint \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{\ell} = \int \boldsymbol{M}_1 \cdot \mathrm{d}\boldsymbol{\ell} = \frac{\mu - \mu_0}{\mu + \mu_0} I$$

3.5（磁场无源性）

- 轴对称磁场 $(\partial B_\phi/\partial\phi = 0)$，已知在原点附近的磁场 z 分量，求 r 分量

$$\nabla\cdot\boldsymbol{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) - 2Cz = 0$$

由此可得 $B_r = Czr$.

3.7（均匀电流圆柱导体的磁矢势）

- $\boldsymbol{A} \propto \int \boldsymbol{J}/r \,\mathrm{d}V'$, 由 $\boldsymbol{J}$ 只有 z 分量可得 $\boldsymbol{A}$ 只有 z 分量。
- 定解问题:
  - $\nabla^2 \boldsymbol{A}_1 = -\mu_0 J\hat{\boldsymbol{z}}\ (r < a);\quad \nabla^2 \boldsymbol{A}_2 = 0\ (r > a)$
  - $r = 0$, $\boldsymbol{A}_1$ 有限
  - 边界条件: $r = a:\ \boldsymbol{A}_1 = \boldsymbol{A}_2 = 0,\ \hat{\boldsymbol{r}} \times (\frac{1}{\mu}\nabla \times \boldsymbol{A}_2 - \frac{1}{\mu_0}\nabla \times \boldsymbol{A}_1) = 0$
- 从电流分布的对称性可得 $\boldsymbol{A}_1 = A_1\hat{\boldsymbol{z}} = A_1(r)\hat{\boldsymbol{z}}$, 由边界条件可得外部矢势 $\boldsymbol{A}_2 = A_2(r)\hat{\boldsymbol{z}}$, 由此可得

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}A_1}{\mathrm{d}r}\right) = -\mu_0 J, \quad \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}A_2}{\mathrm{d}r}\right) = 0$$

解得（积分常数由边界条件确定）

$$\boldsymbol{A}_1 = \frac{\mu_0}{4}(a^2 - r^2)\boldsymbol{J}\,, \quad \boldsymbol{A}_2 = \frac{\mu a^2}{2}\ln\frac{a}{r}\boldsymbol{J}\,.$$

3.9（均匀介质球在均匀外磁场中的磁化）

- 没有自由电流，可以采用磁标势求解（类似于静电势）
- 定解问题：
  - $\nabla^2\varphi_1 = 0\ (R < R_0);\quad \nabla^2\phi_2 = 0\ (R > R_0)$
  - $R = 0, \varphi_1$ 有限；$R \to \infty, \varphi_2 \to -H_0 R\cos\theta$
  - 边界条件：$R = R_0, \varphi_1 = \varphi_2, \mu\dfrac{\partial\varphi_1}{\partial R} = \mu_0\dfrac{\partial\varphi_2}{\partial R}$.
- 通解：

$$\varphi_1 = \sum_n a_n R^n P_n\ ,\quad \varphi_2 = -H_0 R P_1 + \sum_n \frac{b_n}{R^{n+1}} P_n$$

- 带入边界条件解出各系数，结果为：

$$\varphi_1 = -\frac{3\mu_0}{\mu + 2\mu_0}\boldsymbol{H}_0\cdot\boldsymbol{R}\ ,\quad \varphi_2 = -\boldsymbol{H}_0\cdot\boldsymbol{R} + \frac{(\mu - \mu_0)R_0^3}{(\mu + 2\mu_0)R^3}\boldsymbol{H}_0\cdot\boldsymbol{R}$$

3.10（均匀介质空心球的磁屏蔽作用）

- 没有自由电流，可以采用磁标势求解（类似于静电势）
- 定解问题：

$$\varphi_1 = \sum_n a_n R^n P_n \quad (R < R_1)$$

$$\varphi_2 = \sum_n \left(b_n R^n + \frac{c_n}{R^{n+1}} \right) P_n \quad (R_1 < R < R_2)$$

$$\varphi_3 = -H_0 R \cos\theta + \sum_n \frac{d_n}{R^{n+1}} P_n \quad (R > R_3)$$

- 边界条件：

$$R = R_1: \quad \varphi_1 = \varphi_2\,, \quad \mu_0 \frac{\partial \varphi_1}{\partial R} = \mu \frac{\partial \varphi_2}{\partial R}$$

$$R = R_2: \quad \varphi_2 = \varphi_3\,, \quad \mu \frac{\partial \varphi_2}{\partial R} = \mu_0 \frac{\partial \varphi_3}{\partial R}\ .$$

3.10（均匀介质球壳的磁屏蔽作用）

- 解得空腔内的标势 φ_1 和磁场为

$$\varphi_1 = a_1 R\cos\theta\ , \quad \boldsymbol{B}_1 = -\mu_0\nabla\varphi_1 = -\mu_0 a_1\hat{\boldsymbol{z}}\ ,$$

其中

$$a_1 = -H_0 \times \left\{\frac{2(\mu-\mu_0)^2}{9\mu\mu_0}\left[\frac{(\mu+2\mu_0)(2\mu+\mu_0)}{2(\mu-\mu_0)^2} - \left(\frac{R_1}{R_2}\right)^3\right]\right\}^{-1}$$

介质的磁导率 μ 越大，a_1 越弱，$\boldsymbol{B}_1$ 越弱，球壳对外磁场的屏蔽作用越显著。

- David Griffiths, *Introduction to Electrodynamics*, the 4th edition, P245, Example 5.11

It might seem natural to set the polar axis along $\boldsymbol{\omega}$, but in fact the integration is easier if we let $\mathbf{r}$ lie on the z axis, so that $\boldsymbol{\omega}$ is tilted at an angle ψ. We may as well orient the x axis so that $\boldsymbol{\omega}$ lies in the xz plane, as shown in Fig. 5.46. According to Eq. 5.66,

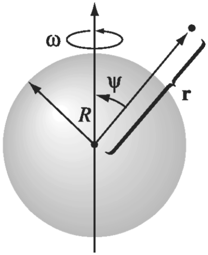

FIGURE 5.45

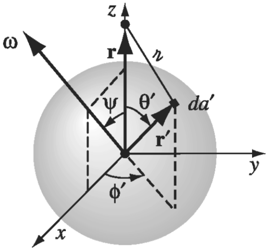

FIGURE 5.46

3.13（匀速转动的均匀带电薄球壳的磁场）

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}} \, da',$$

where $\mathbf{K} = \sigma \mathbf{v}$, $\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$, and $da' = R^2 \sin\theta' \, d\theta' \, d\phi'$. Now the velocity of a point $\mathbf{r}'$ in a rotating rigid body is given by $\boldsymbol{\omega} \times \mathbf{r}'$; in this case,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega \left[-\left(\cos\psi \sin\theta' \sin\phi'\right) \hat{\mathbf{x}} + \left(\cos\psi \sin\theta' \cos\phi' - \sin\psi \cos\theta'\right) \hat{\mathbf{y}} \right.$$

$$\left. + \left(\sin\psi \sin\theta' \sin\phi'\right) \hat{\mathbf{z}} \right].$$

Notice that each of these terms, save one, involves either $\sin\phi'$ or $\cos\phi'$. Since

$$\int_0^{2\pi} \sin\phi' \, d\phi' = \int_0^{2\pi} \cos\phi' \, d\phi' = 0,$$

such terms contribute nothing. There remains

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2} \left(\int_0^{\pi} \frac{\cos\theta' \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \, d\theta' \right) \hat{\mathbf{y}}.$$

3.13（匀速转动的均匀带电薄球壳的磁场）

Letting $u \equiv \cos\theta'$, the integral becomes

$$\int_{-1}^{+1} \frac{u}{\sqrt{R^2+r^2-2Rru}}\,du = -\frac{(R^2+r^2+Rru)}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\,\Bigg|_{-1}^{+1}$$
$$= -\frac{1}{3R^2r^2}\Big[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\Big].$$

If the point $\mathbf{r}$ lies *inside* the sphere, then $R > r$, and this expression reduces to $(2r/3R^2)$; if $\mathbf{r}$ lies *outside* the sphere, so that $R < r$, it reduces to $(2R/3r^2)$. Noting that $(\boldsymbol{\omega} \times \mathbf{r}) = -\omega r \sin\psi\,\hat{\mathbf{y}}$, we have, finally,

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\boldsymbol{\omega}\times\mathbf{r}), & \text{for points } inside \text{ the sphere,} \\ \dfrac{\mu_0 R^4\sigma}{3r^3}(\boldsymbol{\omega}\times\mathbf{r}), & \text{for points } outside \text{ the sphere.} \end{cases} \tag{5.68}$$

Having evaluated the integral, I revert to the "natural" coordinates of Fig. 5.45, in which $\boldsymbol{\omega}$ coincides with the z axis and the point $\mathbf{r}$ is at (r, θ, ϕ):

$$\mathbf{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\boldsymbol{\phi}}, & (r \le R), \\ \dfrac{\mu_0 R^4\omega\sigma}{3}\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}, & (r \ge R). \end{cases} \tag{5.69}$$

3.13（匀速转动的均匀带电薄球壳的磁场）

Curiously, the field inside this spherical shell is *uniform*:

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{2}{3}\mu_0\sigma R\omega\,\hat{\mathbf{z}} = \frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}. \quad (5.70)$$