CMB physics

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2. Primordial Anisotropy

Key concept

2.1 Linear perturbation theory

- Adiabatic curvature perturbation
- Conformal Newtonian gauge
- Synchronous gauge
- Acoustic oscillation
- Damping
- Thomson scattering
- LoS projection
- Boltzmann Eq.

8. Relativistic perturbation theory

unlike Newtonian theory, GR starts from the metric field

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

$$ds^2 = a^2(\tau)\left[d\tau^2 - \delta_{ij}dx^i dx^j\right]$$

$$ds^2 = a^2(\tau)\left[(1+2A)d\tau^2 - 2B_i dx^i d\tau - (\delta_{ij} + h_{ij})dx^i dx^j\right]$$

decompose $\delta g_{\mu\nu}$ into scalar, vector, tensor according to SO(3) rotation group

$\hat{E}^i{}_i = 0$ (traceless) $\qquad \partial^i \hat{E}_{ij} = 0$ (transverse)

$$B_i = \underbrace{\partial_i B}_{\text{scalar}} + \underbrace{\hat{B}_i}_{\text{vector}}$$

$$h_{ij} = \underbrace{2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E}_{\text{scalar}} + \underbrace{2\partial_{(i}\hat{E}_{j)}}_{\text{vector}} + \underbrace{2\hat{E}_{ij}}_{\text{tensor}}$$

$$\partial_{\langle i}\partial_{j\rangle}E \equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E$$

$$\partial_{(i}\hat{E}_{j)} \equiv \frac{1}{2}\left(\partial_i\hat{E}_j + \partial_j\hat{E}_i\right).$$

- *scalars:* A, B, C, E
- *vectors:* $\hat{B}_i, \hat{E}_i$
- *tensors:* $\hat{E}_{ij}$

$$\delta g_{\mu\nu} = \begin{Bmatrix} 00 & 01 & 02 & 03 \\ & 11 & 12 & 13 \\ & & 22 & 23 \\ & & & 33 \end{Bmatrix}$$

$$d.o.f. = \begin{Bmatrix} s = 4 \\ v = 6 - 2 = 4 \\ t = 6 - 1 - 3 = 2 \end{Bmatrix}$$

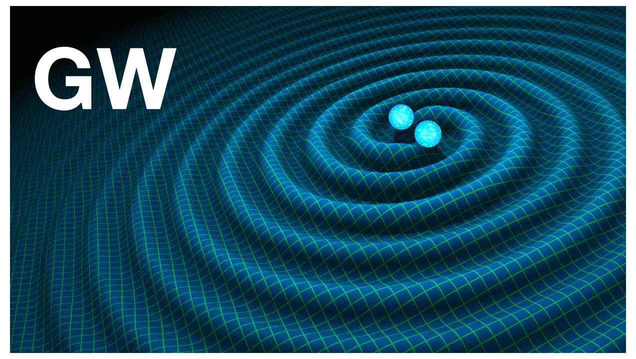
$\nabla^\mu G_{\mu\nu} = 0$ (Bianchi identity)

choosing a coordinate frame $\{x^\mu\}$

10 - 4 - 4 = 2

pure gravitational d.o.f.

$\hat{E}_{ij}$ tensor pert.



scalar & vector pert. are just the reaction of the gravity to matter sector

e.g. $\nabla^2\Phi = 4\pi G\delta\rho$

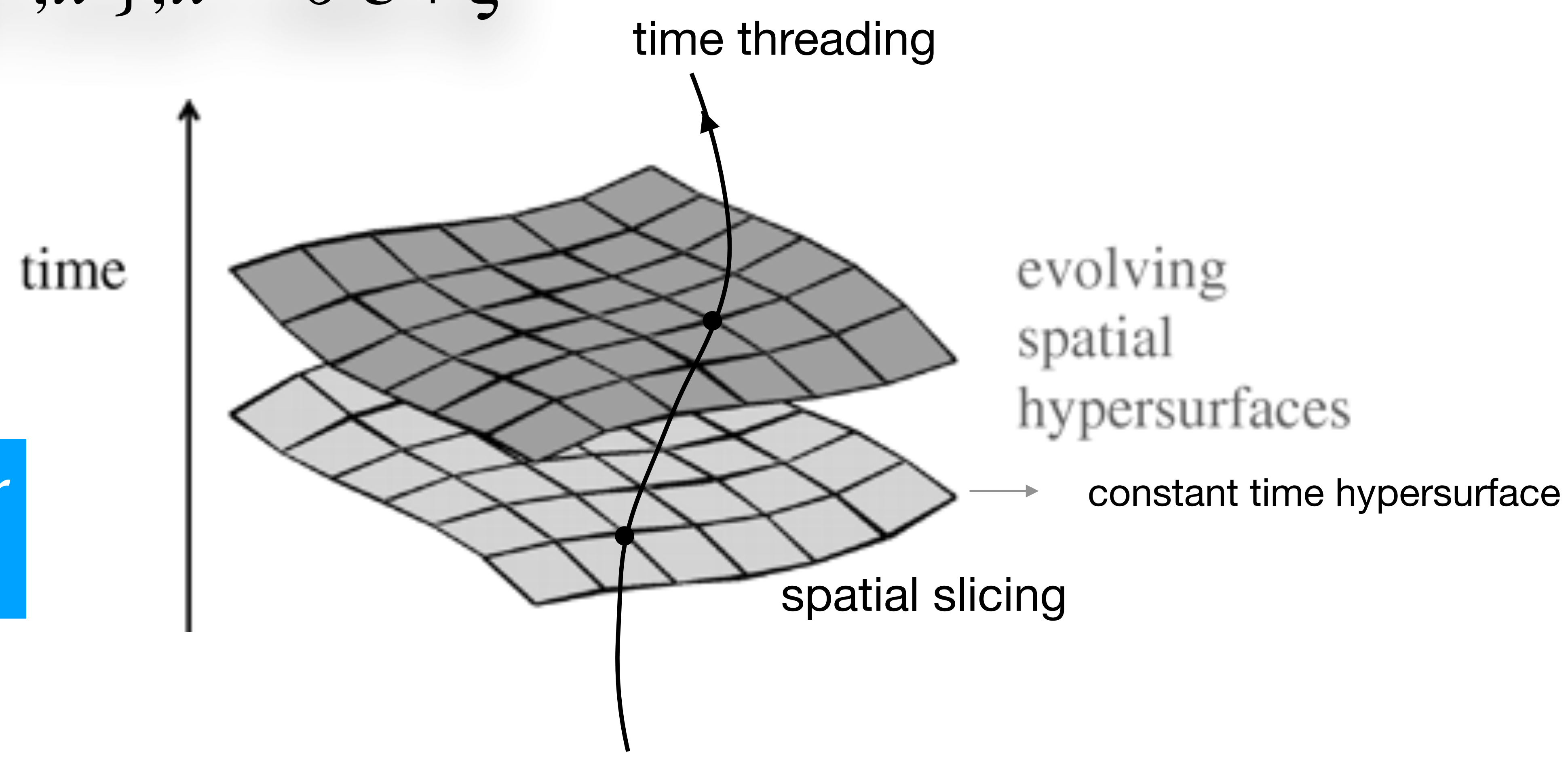
However, tensor pert. can exist even in the matter vacuum!

A pure gravitational phenomena.

gauge fixing $\Longleftrightarrow$ choosing coordinate $x^{\mu} = \{x^0, x^i\}, x^i = \partial^i \varepsilon + \xi^i$

$$d.o.f. = \begin{Bmatrix} s = 4-2 = 2 \\ v = 4-2 = 2 \\ t = 2 \end{Bmatrix}$$

Theorem: At linear order, scalar/vector/tensor pert. are completely decoupled.



In FRWL background evolution, vector pert. only have decaying mode, so cosmologically irrelevant.

From now on, we only consider scalar & tensor pert. Comparing scalar & tensor pert, signal from scalar > tensor

Mathematically, scalar is more complicated than tensor. Because tensor is gauge invariant, why?

[gauge/coordinate transformation, does not involve tensor. Hence, tensor mode is free of gauge issue]

$$d.o.f. = \begin{Bmatrix} s = 4-2 = 2 \\ \cancel{v = 4-2 = 2} \\ \cancel{t = 2} \end{Bmatrix}$$

scalar mode gauge fixing

2 commonly used gauge

- *Newtonian gauge.*—The choice

$$B = E = 0\,,$$

gives the metric

$$\mathrm{d}s^2 = a^2(\tau)\left[(1+2\Psi)\mathrm{d}\tau^2 - (1-2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]\,.$$

- synchronous gauge (a frame co-moving with cosmic fluid)

$$ds^2 = a^2[d\tau^2 - (\delta_{ij} + h_{ij})dx^i dx^j] \qquad A = B = 0 \qquad h_{ij} = 2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E$$

gauge transformation

Consider the coordinate transformation

$$X^\mu \mapsto \tilde{X}^\mu \equiv X^\mu + \xi^\mu(\tau, \boldsymbol{x}) \, ,$$

$$\xi^0 \equiv T \, , \quad \xi^i \equiv L^i = \partial^i L + \hat{L}^i$$

$$\mathrm{d}s^2 = g_{\mu\nu}(X)\mathrm{d}X^\mu \mathrm{d}X^\nu = \tilde{g}_{\alpha\beta}(\tilde{X})\mathrm{d}\tilde{X}^\alpha \mathrm{d}\tilde{X}^\beta$$

$$g_{\mu\nu}(X) = \frac{\partial \tilde{X}^\alpha}{\partial X^\mu}\frac{\partial \tilde{X}^\beta}{\partial X^\nu}\tilde{g}_{\alpha\beta}(\tilde{X})$$

[Pb4.]

ref: Baumann lecture eq. (4.2.48)~(4.2.60)

$$A \mapsto A - T' - \mathcal{H}T \, ,$$
$$B \mapsto B + T - L' \, ,$$
$$C \mapsto C - \mathcal{H}T - \frac{1}{3}\nabla^2 L \, ,$$
$$E \mapsto E - L \, ,$$

$$\hat{B}_i \mapsto \hat{B}_i - \hat{L}'_i \, ,$$
$$\hat{E}_i \mapsto \hat{E}_i - \hat{L}_i \, ,$$

gauge-inv

$$\hat{E}_{ij} \mapsto \hat{E}_{ij}$$

gauge-inv variables

(Bardeen potential)

$$\Psi \equiv A + \mathcal{H}(B - E') + (B - E')'$$
$$\Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E \, .$$

(check)

A & C in conformal Newtonian gauge, equals Ψ & Φ respectively.

$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu + \delta T^\mu{}_\nu \qquad \bar{T}^\mu{}_\nu = (\bar{\rho} + \bar{P})\bar{U}^\mu \bar{U}_\nu - \bar{P}\delta^\mu_\nu$$

$\bar{U}_\mu = a\delta^0_\mu, \ \bar{U}^\mu = a^{-1}\delta^\mu_0$ for a comoving observer

perfect fluid: no energy dissipation, can not conduct heat

perfect fluid does not exist in real life, but compare with honey, water can be treated as perfect fluid.

$$g_{\mu\nu}U^\mu U^\nu = 1$$

$$\delta T^\mu{}_\nu = (\delta\rho + \delta P)\bar{U}^\mu \bar{U}_\nu + (\bar{\rho} + \bar{P})(\delta U^\mu \bar{U}_\nu + \bar{U}^\mu \delta U_\nu) - \delta P \delta^\mu_\nu - \Pi^\mu{}_\nu$$

shear-viscosity

$$U^\mu = a^{-1}[1 - A, v^i]$$

(deriv)

$$T^{\mu\nu}_{vf} = \rho\, u^\mu u^\nu + (p + p_b)\,\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$p_b = -\zeta\, \nabla_\mu u^\mu$$

bulk-viscosity

$P(\rho), P_b(\nabla u)$ P: describe the ability to do external work, the mount of work only depends on the initial & final config

P_b: internal energy loss, the mount of energy loss depends also on the volume changing velocity

[Pb5.]

$$\delta T^0{}_0 = \delta\rho \,,$$
$$\delta T^i{}_0 = (\bar\rho + \bar P) v^i \,,$$
$$\delta T^0{}_j = -(\bar\rho + \bar P)(v_j + B_j)$$
$$\delta T^i{}_j = -\delta P \delta^i_j - \Pi^i{}_j \,.$$

$$T^\mu{}_\nu(X) = \frac{\partial X^\mu}{\partial \tilde X^\alpha}\frac{\partial \tilde X^\beta}{\partial X^\nu}\tilde T^\alpha{}_\beta(\tilde X)$$

[Pb6.]

$$\delta\rho \mapsto \delta\rho - T\bar\rho' \,,$$
$$\delta P \mapsto \delta P - T\bar P' \,,$$
$$q_i \mapsto q_i + (\bar\rho + \bar P) L_i'$$
$$v_i \mapsto v_i + L_i' \,,$$
$$\Pi_{ij} \mapsto \Pi_{ij} \,.$$

ref: Baumann lecture eq. (4.2.68)~(4.2.73)

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pert. classification

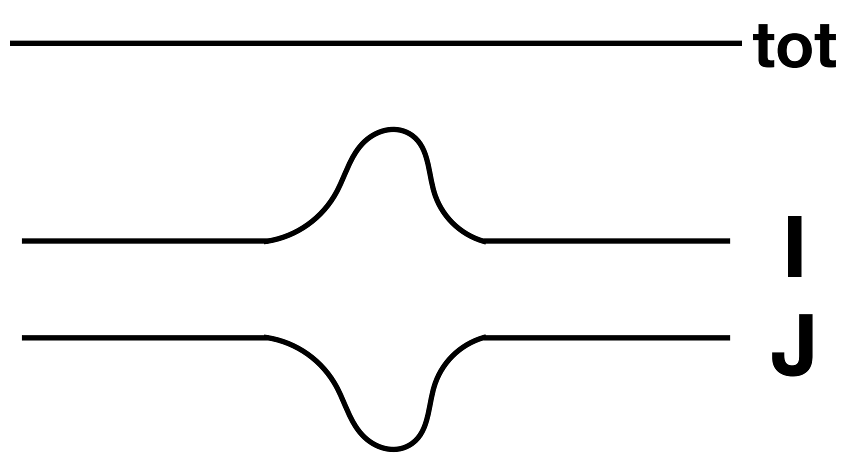
- adiabatic pert. ⟶ time delay $\qquad \delta\rho_I(\tau, \boldsymbol{x}) \equiv \bar\rho_I(\tau + \delta\tau(\boldsymbol{x})) - \bar\rho_I(\tau) = \bar\rho_I' \delta\tau(\boldsymbol{x})$

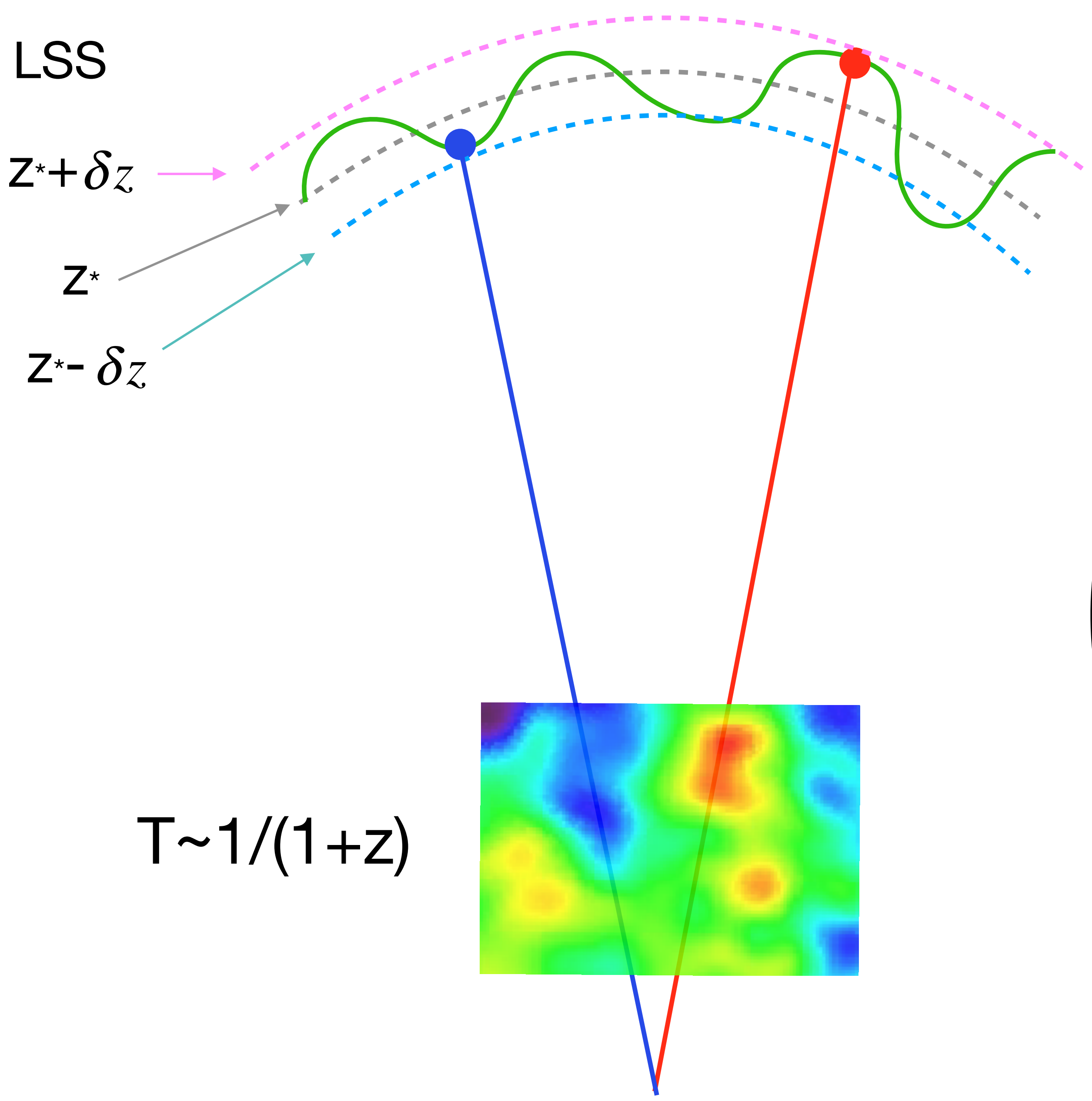
$$\delta\tau = \frac{\delta\rho_I}{\bar\rho_I'} = \frac{\delta\rho_J}{\bar\rho_J'} \quad \text{for all species } I \text{ and } J$$

(not independent!)

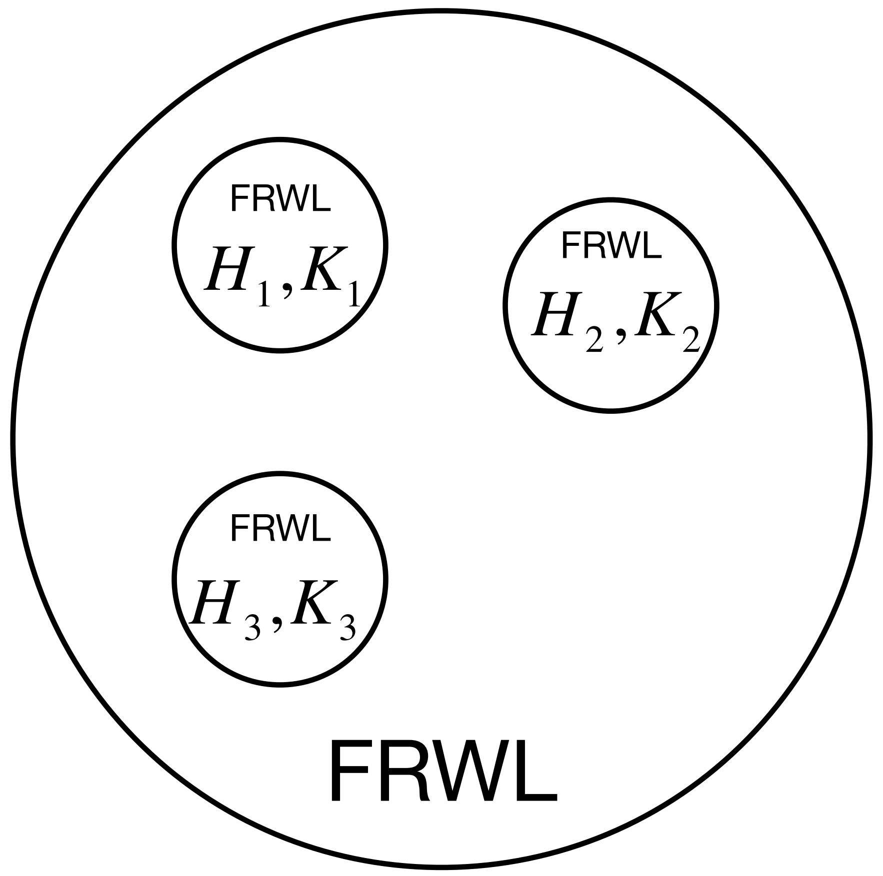
- isocurvature/entropy pert.

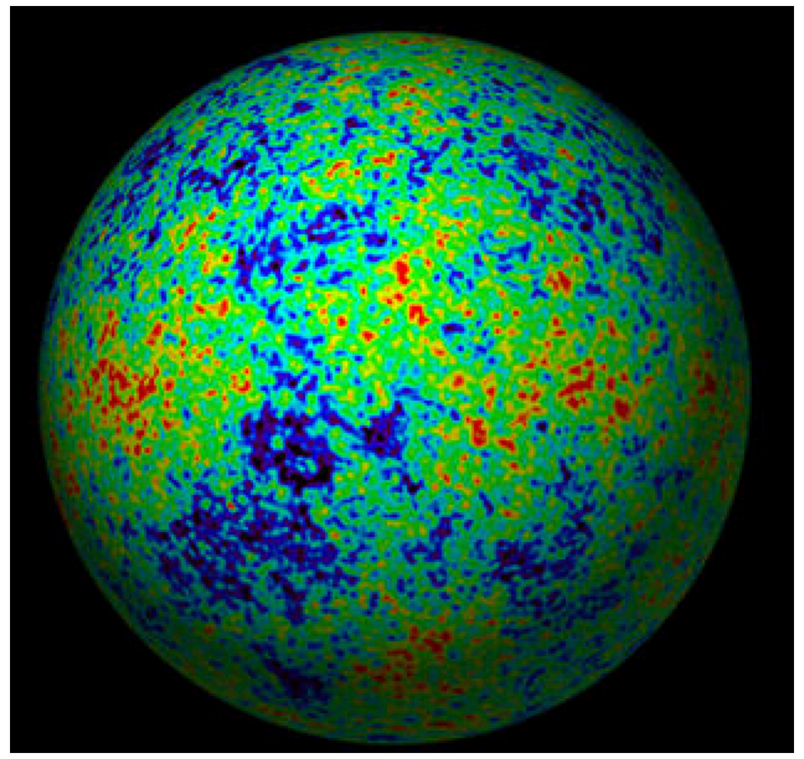
$$\delta P(\rho, s) = \frac{\partial P}{\partial \rho}\delta\rho + \boxed{\frac{\partial P}{\partial s}\delta s}$$





separate universe assumption





Linearised Einstein eq.

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho} \right)$$

[Pb7.]

$$\Gamma^0_{00} = \mathcal{H} + \Psi' \,, \quad \text{(deriv)}$$
$$\Gamma^0_{0i} = \partial_i \Psi \,,$$
$$\Gamma^i_{00} = \delta^{ij} \partial_j \Psi \,,$$
$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} - \left[\Phi' + 2\mathcal{H}(\Phi + \Psi) \right] \delta_{ij}$$
$$\Gamma^i_{j0} = \mathcal{H}\delta^i_j - \Phi' \delta^i_j \,,$$
$$\Gamma^i_{jk} = -2\delta^i_{(j}\partial_{k)}\Phi + \delta_{jk}\delta^{il}\partial_l \Phi \,. \quad \text{(deriv)}$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}$$

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}$$

[Pb8.]

$$R_{00} = -3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi' + \Psi') + 3\Phi'' \,, \quad \text{(deriv)}$$
$$R_{0i} = 2\partial_i \Phi' + 2\mathcal{H}\partial_i \Psi \,,$$
$$R_{ij} = \left[\mathcal{H}' + 2\mathcal{H}^2 - \Phi'' + \nabla^2\Phi - 2(\mathcal{H}' + 2\mathcal{H}^2)(\Phi + \Psi) - \mathcal{H}\Psi' - 5\mathcal{H}\Phi' \right] \delta_{ij} + \partial_i \partial_j (\Phi - \Psi) \,.$$

$$a^2 R = -6(\mathcal{H}' + \mathcal{H}^2) + 2\nabla^2\Psi - 4\nabla^2\Phi + 12(\mathcal{H}' + \mathcal{H}^2)\Psi + 6\Phi'' + 6\mathcal{H}(\Psi' + 3\Phi') \quad \text{(deriv)}$$

[Pb9.]

$$G_{00} = 3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi'$$

$$G_{0i} = 2\partial_i(\Phi' + \mathcal{H}\Psi)$$

$$G_{ij} = -(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} + \left[\nabla^2(\Psi - \Phi) + 2\Phi'' + 2(2\mathcal{H}' + \mathcal{H}^2)(\Phi + \Psi) + 2\mathcal{H}\Psi' + 4\mathcal{H}\Phi' \right] \delta_{ij} + \partial_i \partial_j (\Phi - \Psi) \,. \tag{4.2.134}$$

- *Newtonian gauge.*—The choice

$$B = E = 0 \,,$$

gives the metric

$$ds^2 = a^2(\tau) \left[(1 + 2\Psi) d\tau^2 - (1 - 2\Phi)\delta_{ij} dx^i dx^j \right] \,.$$

$$ds^2 = a^2(\tau)\left[(1+2\Psi)d\tau^2 - (1-2\Phi)\delta_{ij}dx^i dx^j\right] . \quad (4.4.168)$$

In these lectures, we won't encounter situations where anisotropic stress plays a significant role, so we will always be able to set $\Psi = \Phi$.

- The Einstein equations then are

[Pb10.]

$$\begin{aligned} \nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) &= 4\pi G a^2 \delta\rho , \quad \text{(deriv)} & (4.4.169) \\ \Phi' + \mathcal{H}\Phi &= -4\pi G a^2(\bar{\rho} + \bar{P})v , & (4.4.170) \\ \Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi &= 4\pi G a^2 \delta P . & (4.4.171) \end{aligned}$$

neglect time evolution term $\downarrow$ $\Psi = \Phi$

Poisson eq. $\nabla^2\Psi = 4\pi G\delta\rho$

$k < \mathcal{H}$ / $L > 1/\mathcal{H}$

On the cosmic large scale, we do need relativity theory!

On the small scale, Newtonian theory works well!

$k > \mathcal{H}$ / $L < 1/\mathcal{H}$

$\mathcal{H}^{-1} \longrightarrow$ **co-moving Hubble radius**

2. Primary CMB

Key concept

2.2 Boltzmann Eq.

- Spherical projection of the Plane wave
- Compton scattering vs Thompson scattering
- Acoustic oscillation
- Baryon load
- Velocity/Density are out of phase
- Transfer function

T~1GeV

quark -> proton&neutron

T~1MeV

nucleosynthesis: create light element D, He, Li

T~0.1eV

recombination: $p^+ + e^- \to H$

z~(100,1000)

dark ages

z~(6,10)

reionization

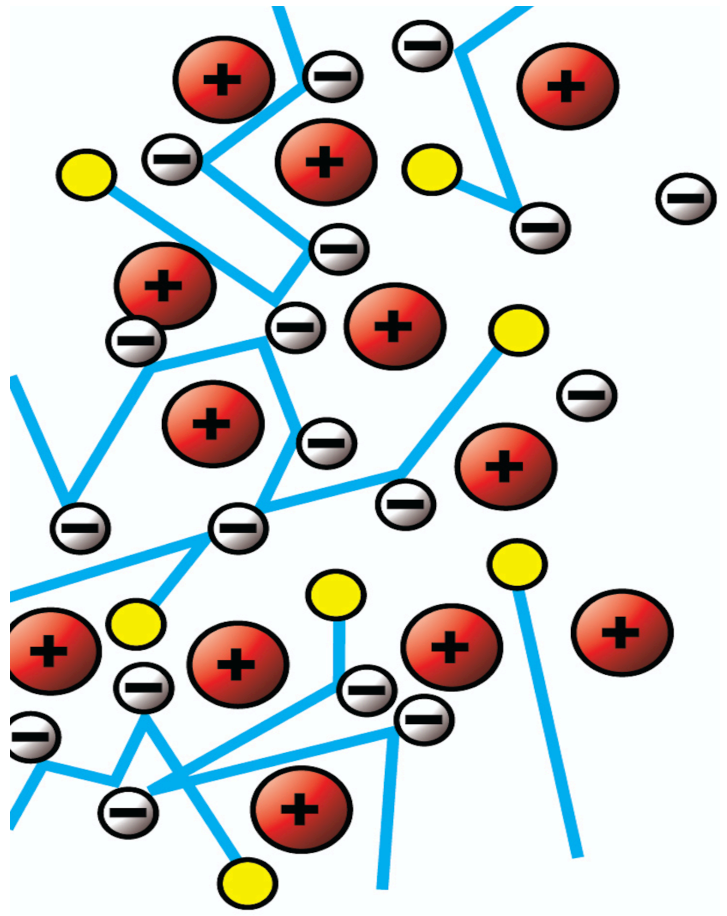
z<2

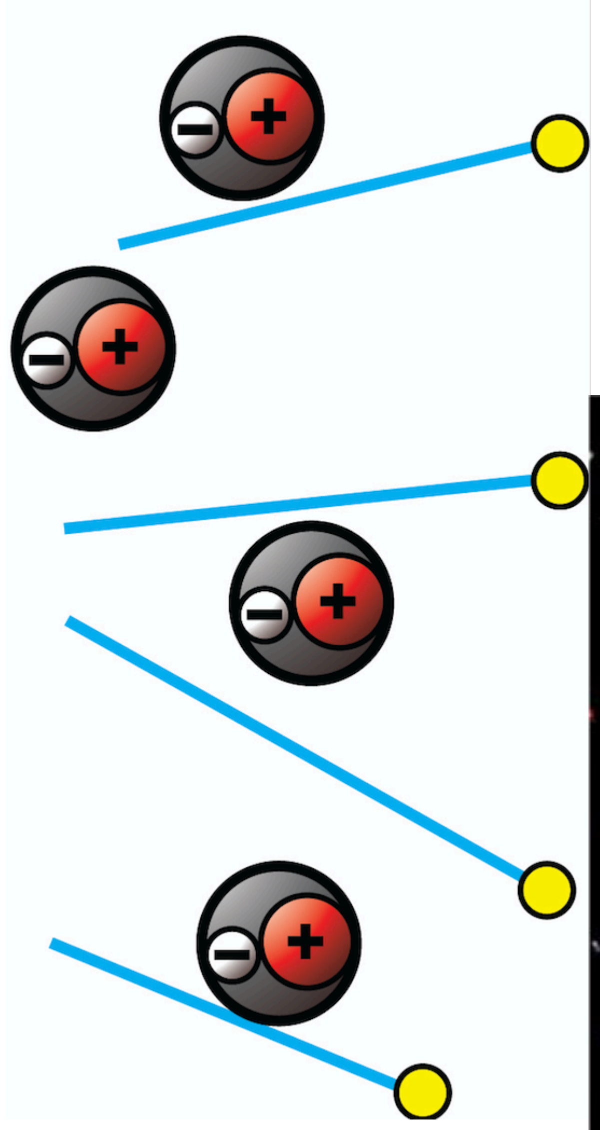
galaxy cluster

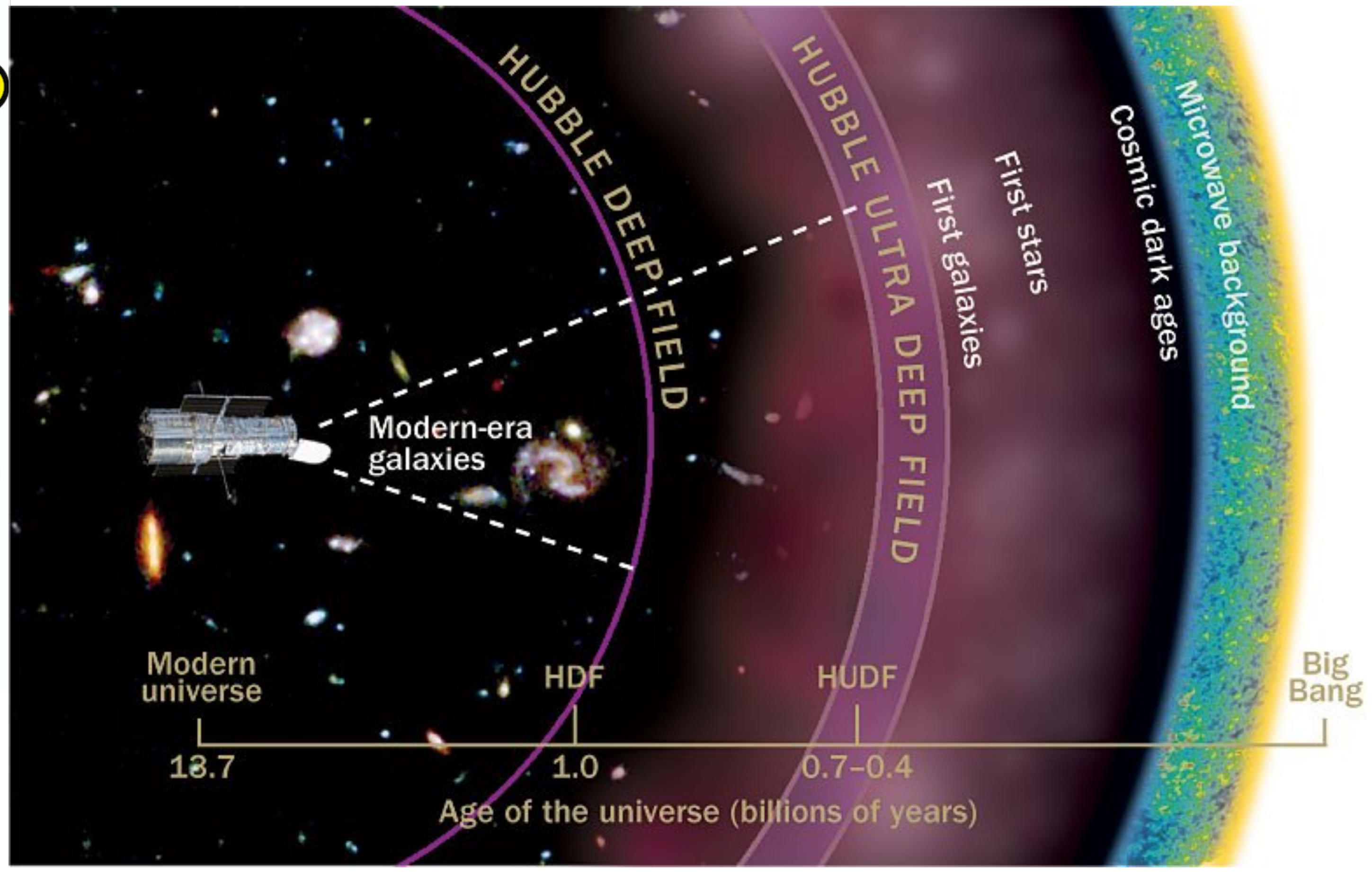
z<1

Dark energy

The faintest light from the very early universe.







A dim view
A map of the universe looks back in time to the cosmic dark ages, the interval between the time when radiation left over from the Big Bang streamed freely into space and when galaxies produced enough ultraviolet light to reionize the universe. Hubble, shown, cannot see that far back.

opaque

transparent

1948 Gamow

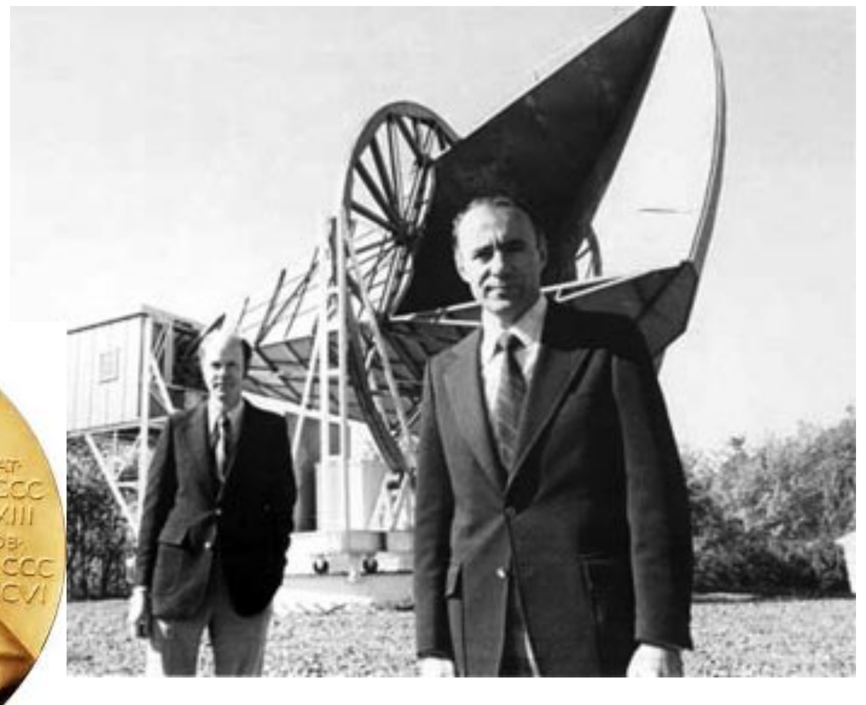
Hot Big Bang

$T_{CMB}=5K$

1964

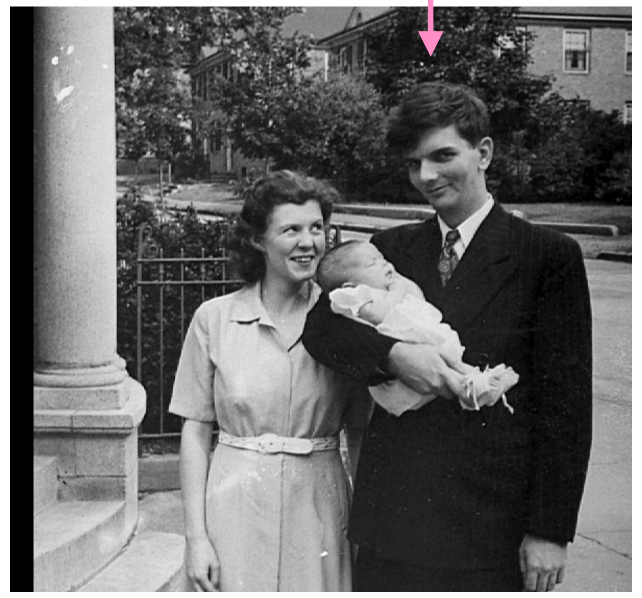
Penzias & Wilson

Nobel prize in Physics 1978



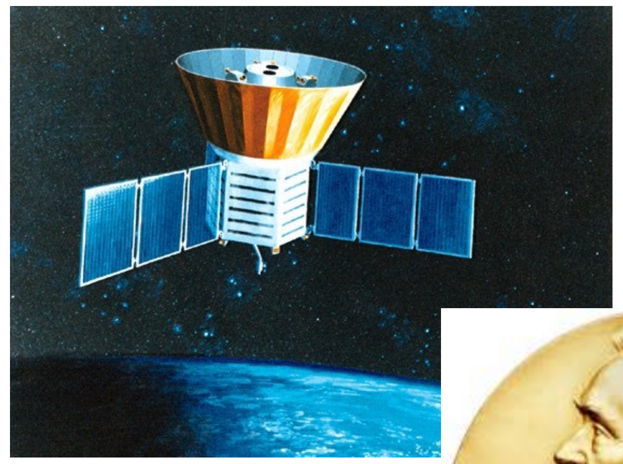
Dicke，Wilkinson，Peebles et. al.

2019



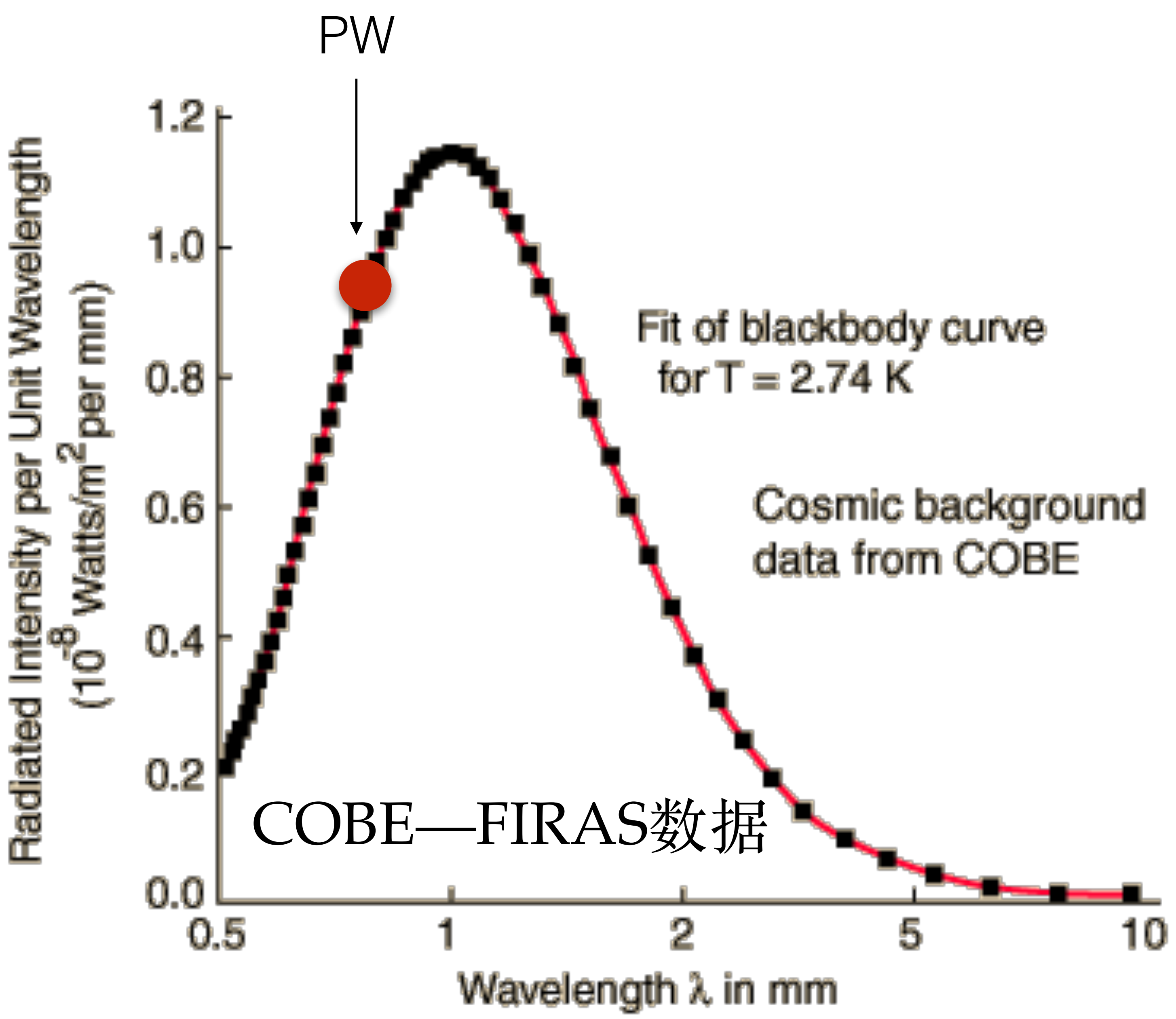
1992

COBE (NASA)

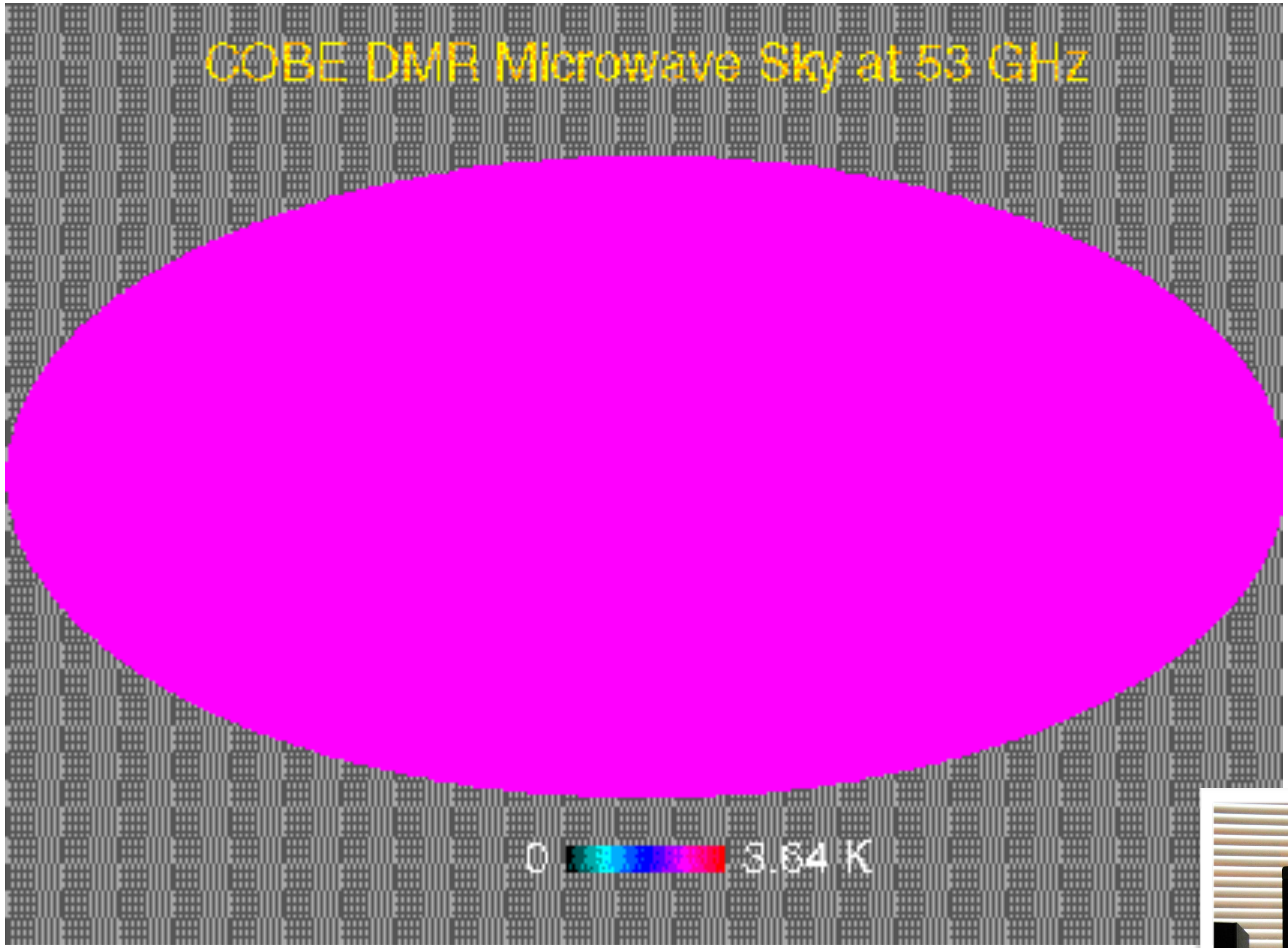


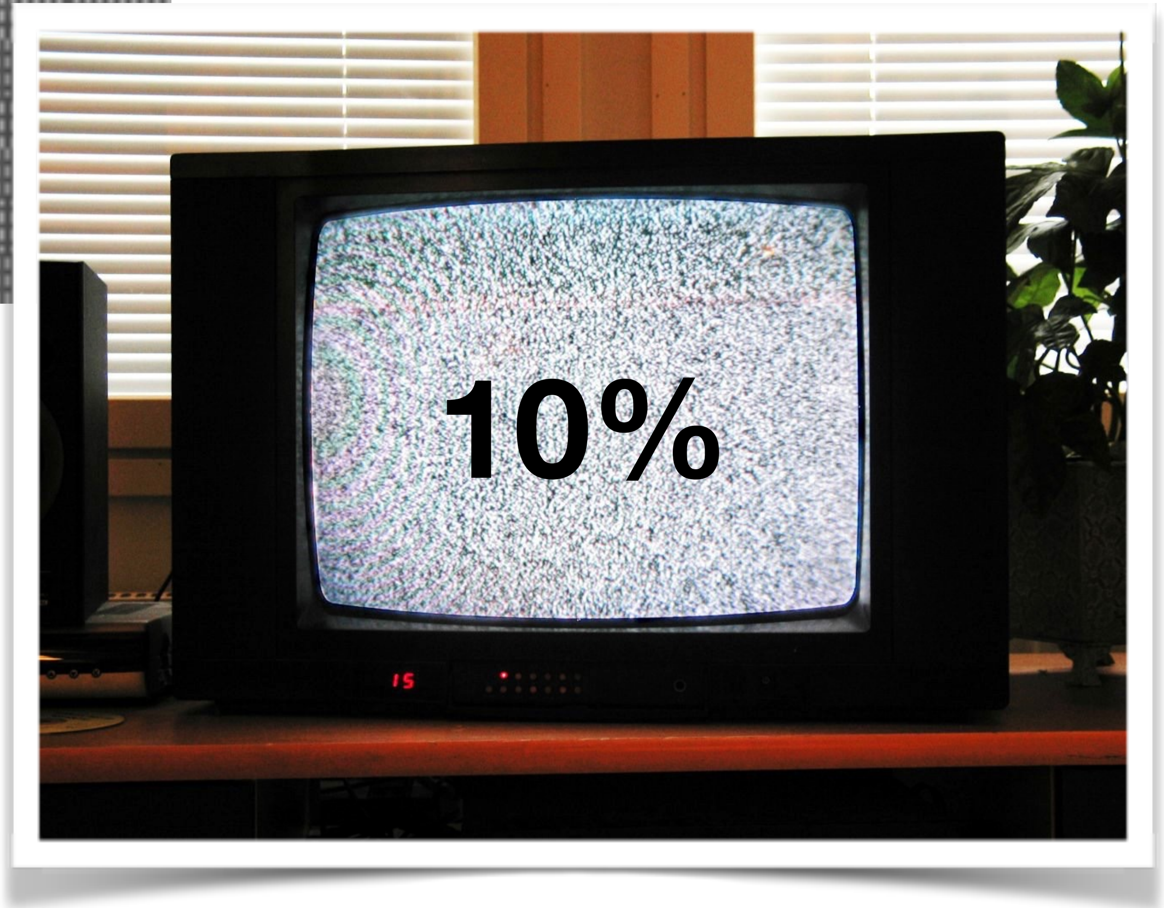
Nobel prize in Physics 2006

Mather & Smoot

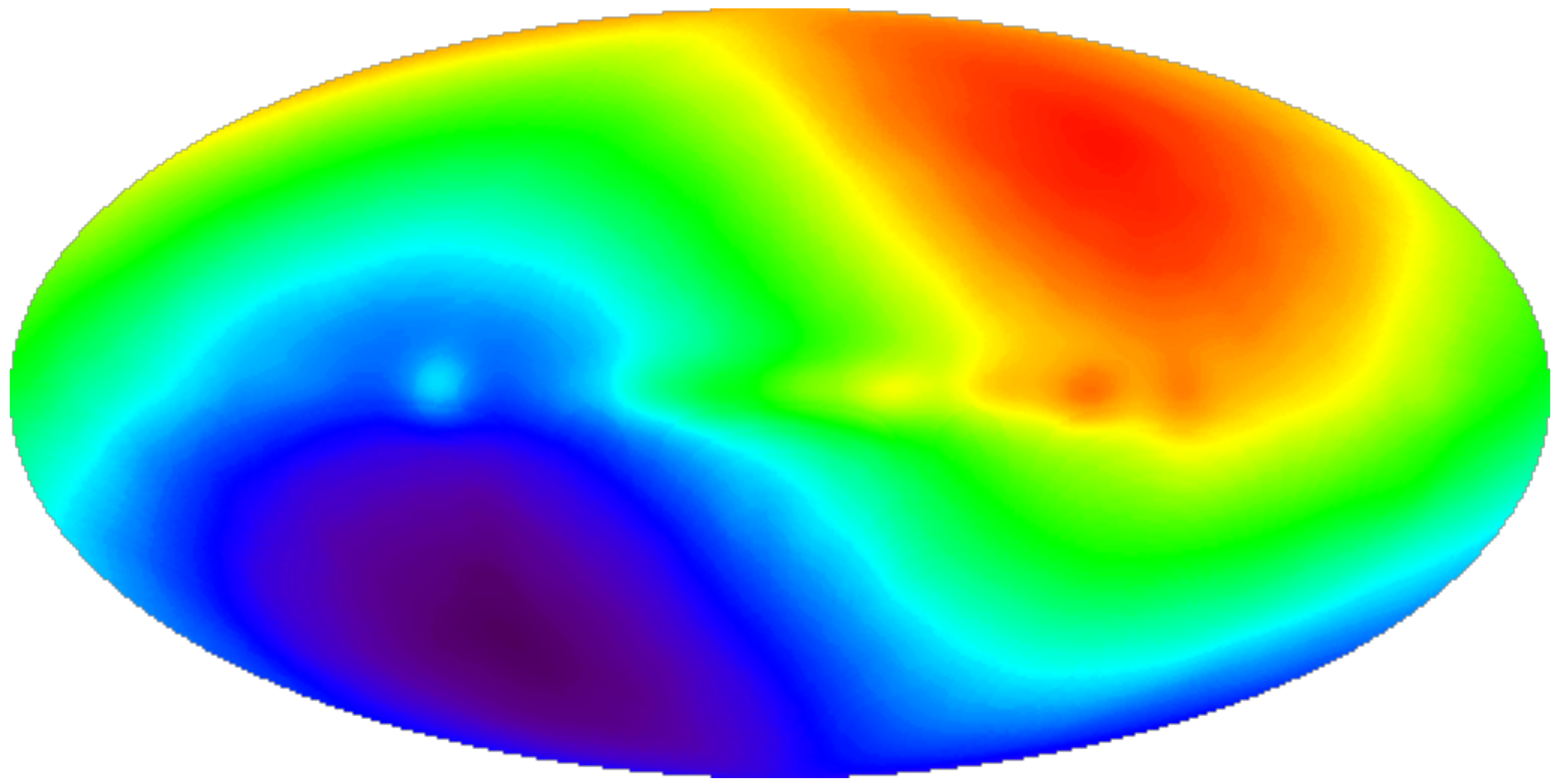


CMB isotropy





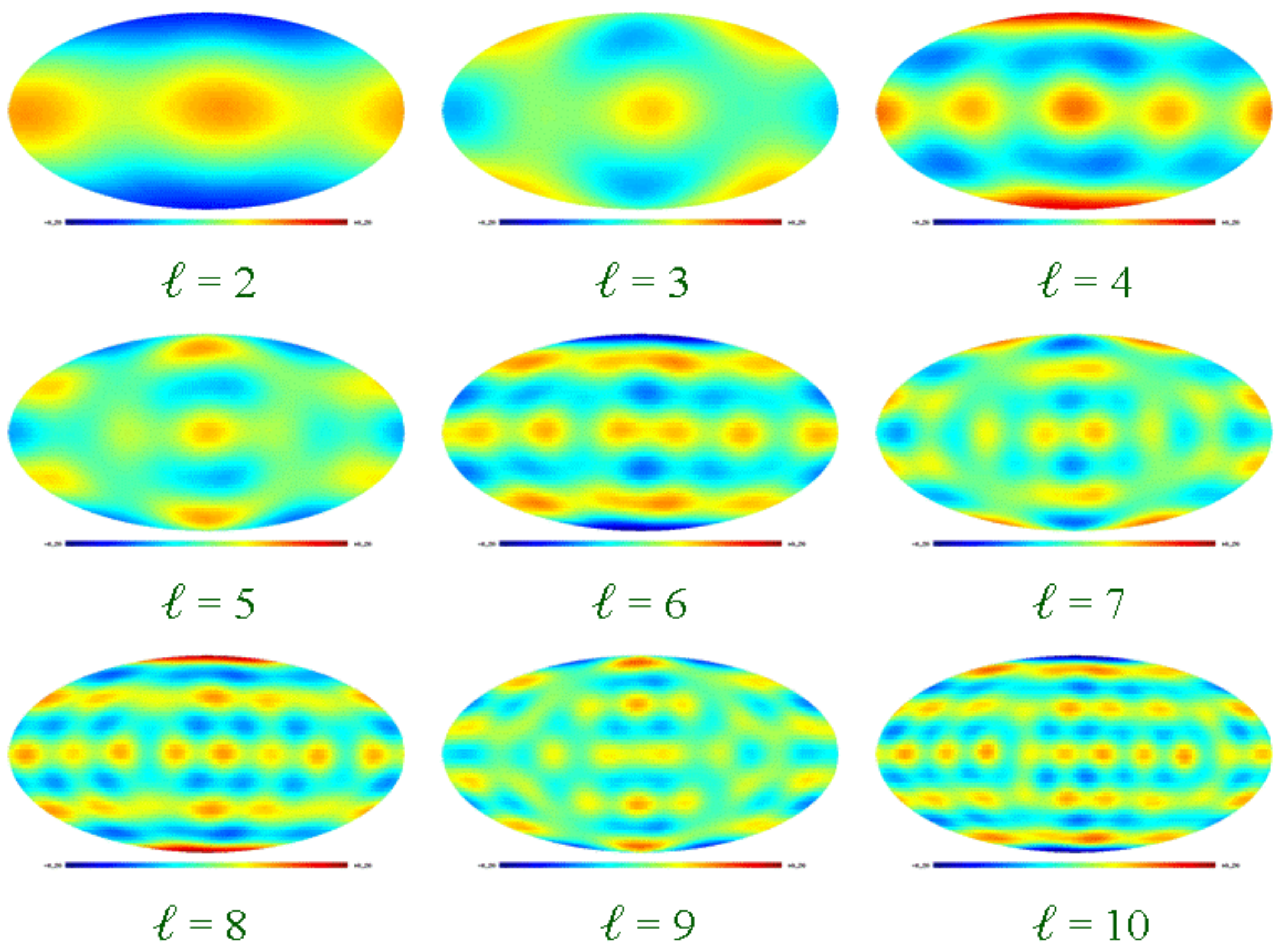
CMB anisotropy

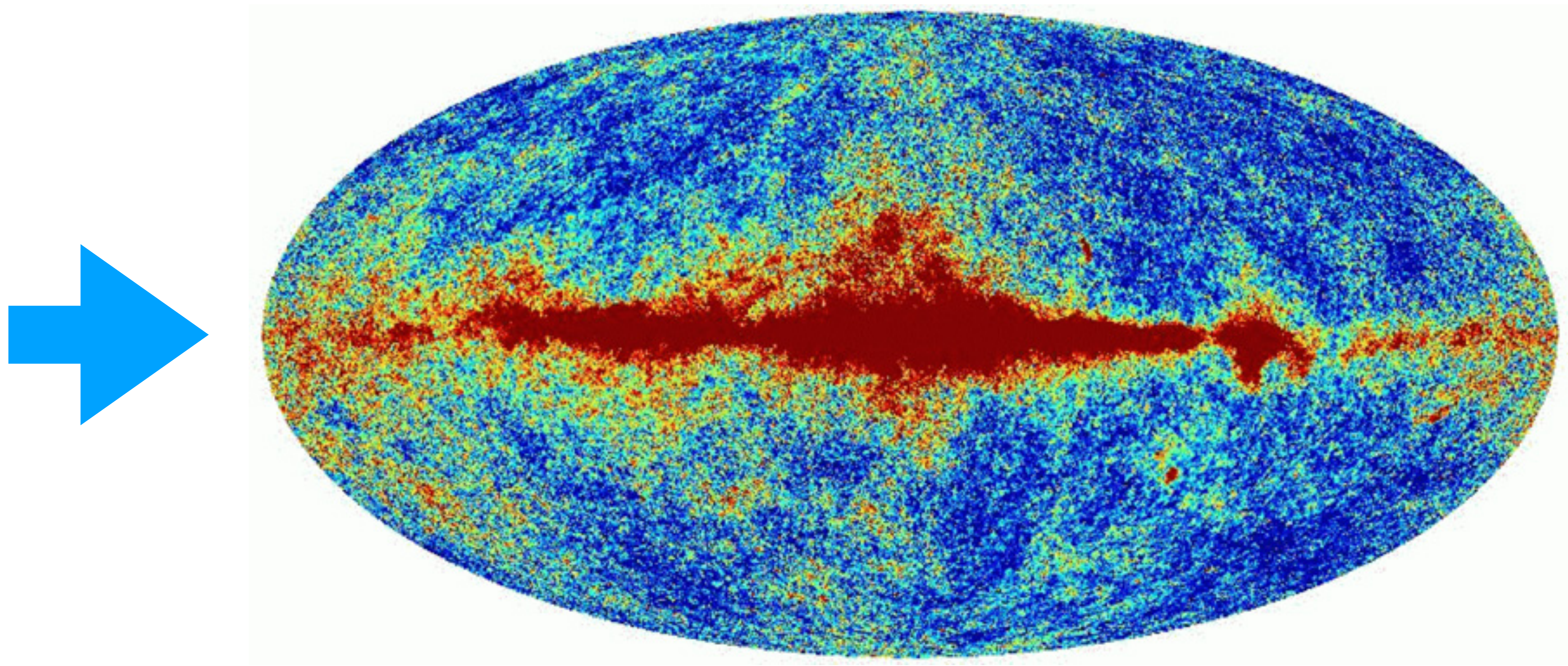


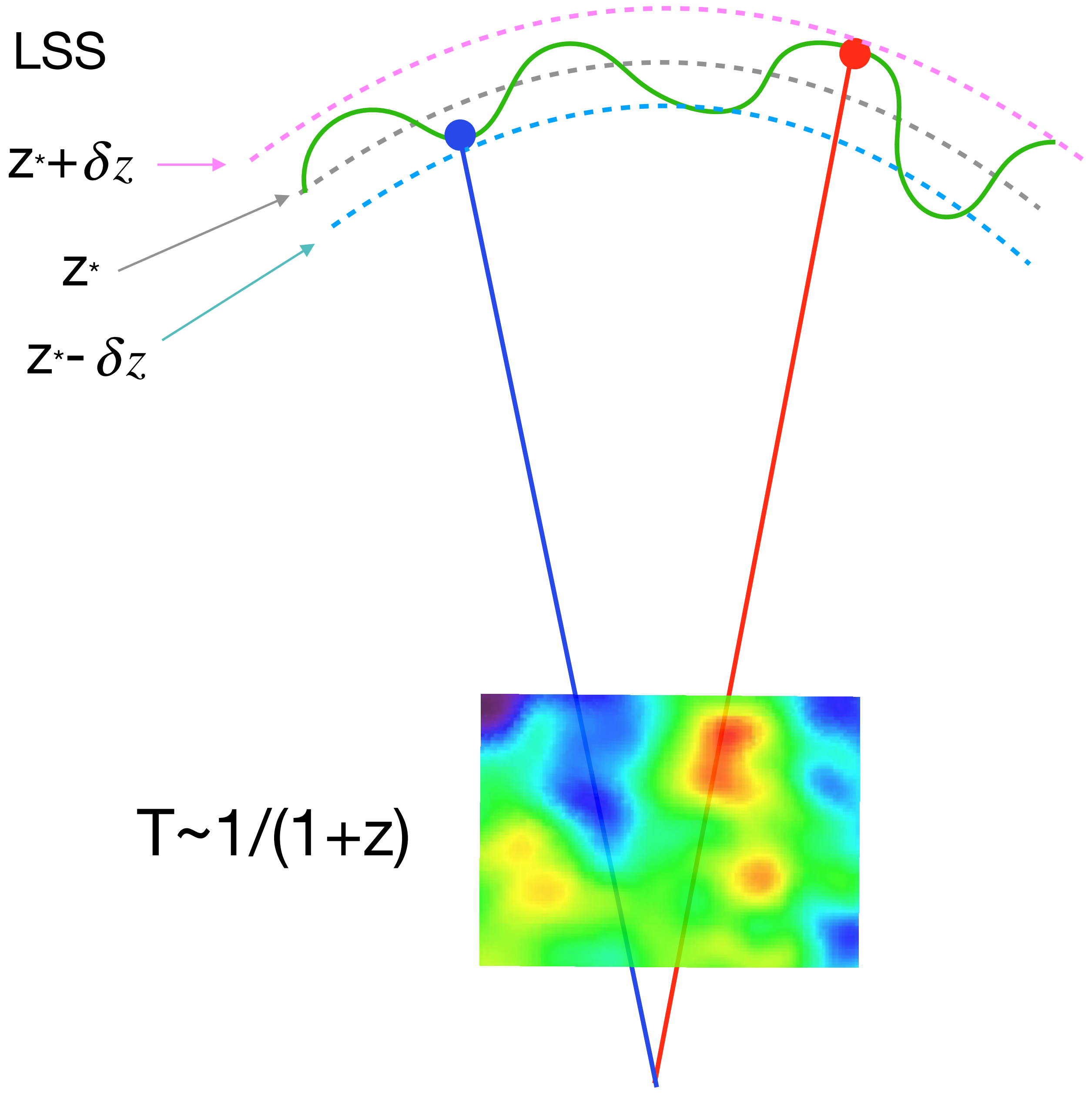
Dipole $\Delta T = 3.353 mK$

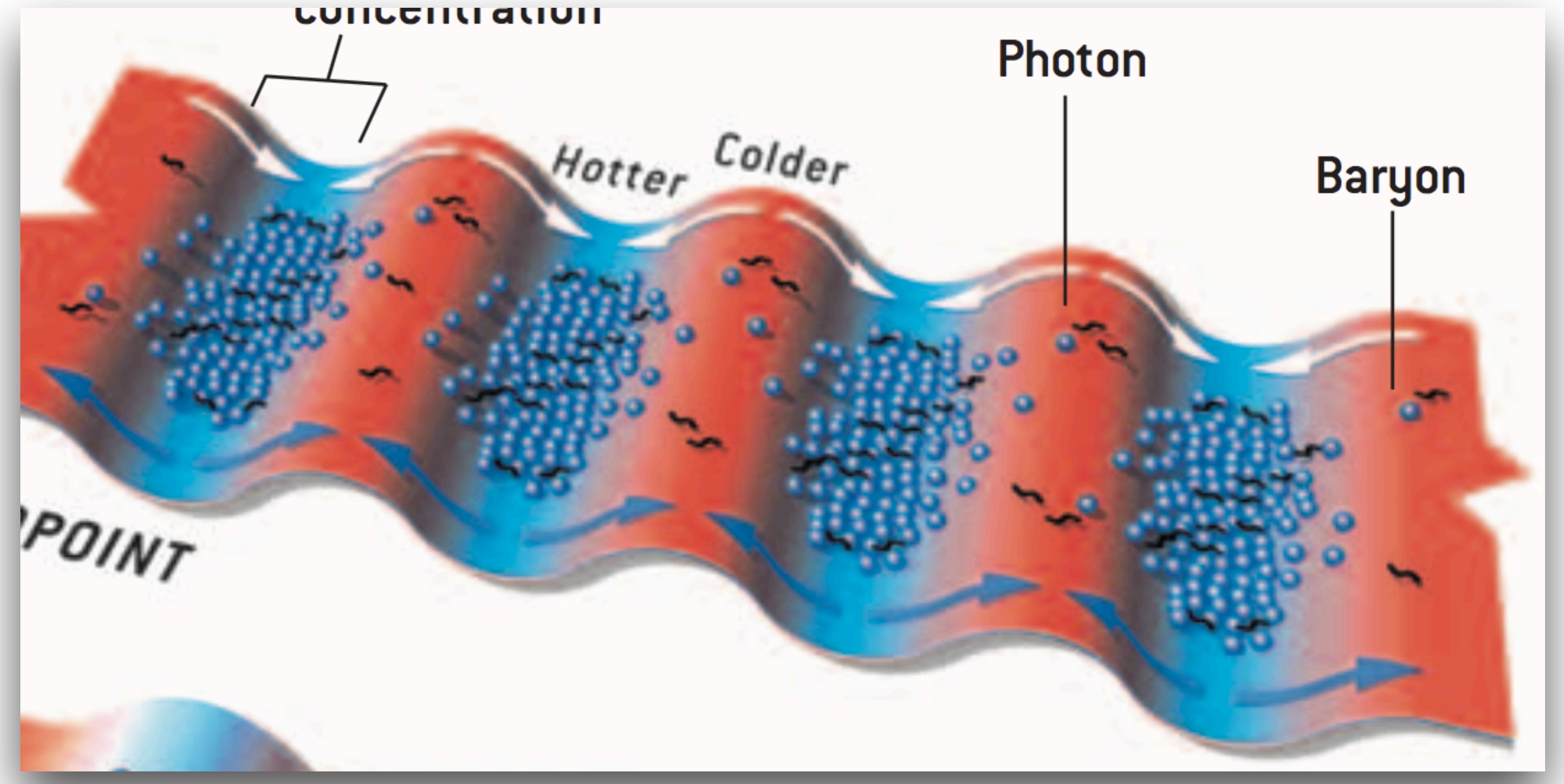
due to relative motion of our earth w.r.t. rest frame of CMB

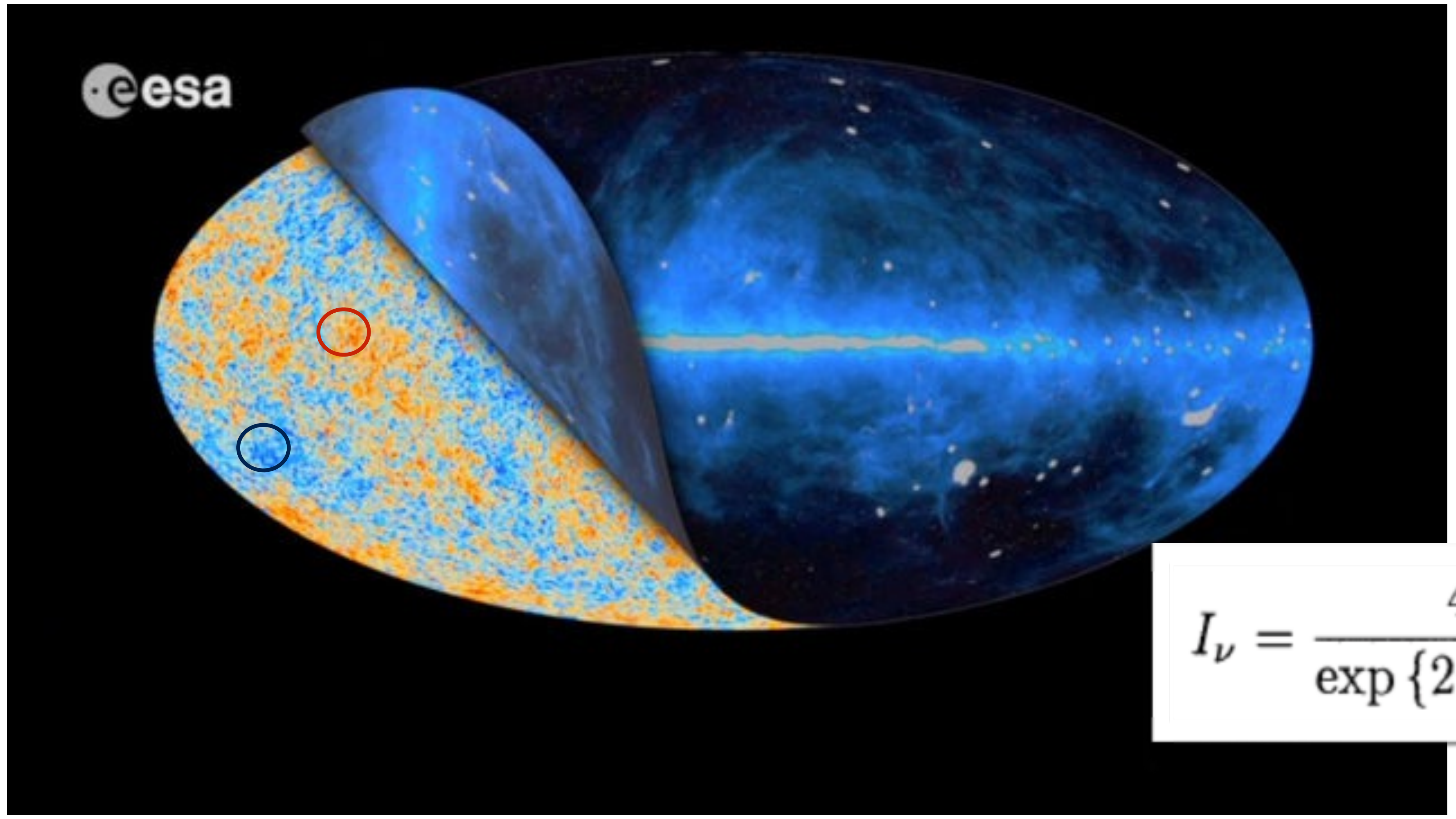
higher multipoles $\Delta T \sim 18 \mu K$ due to primordial gravitational curvature pert.









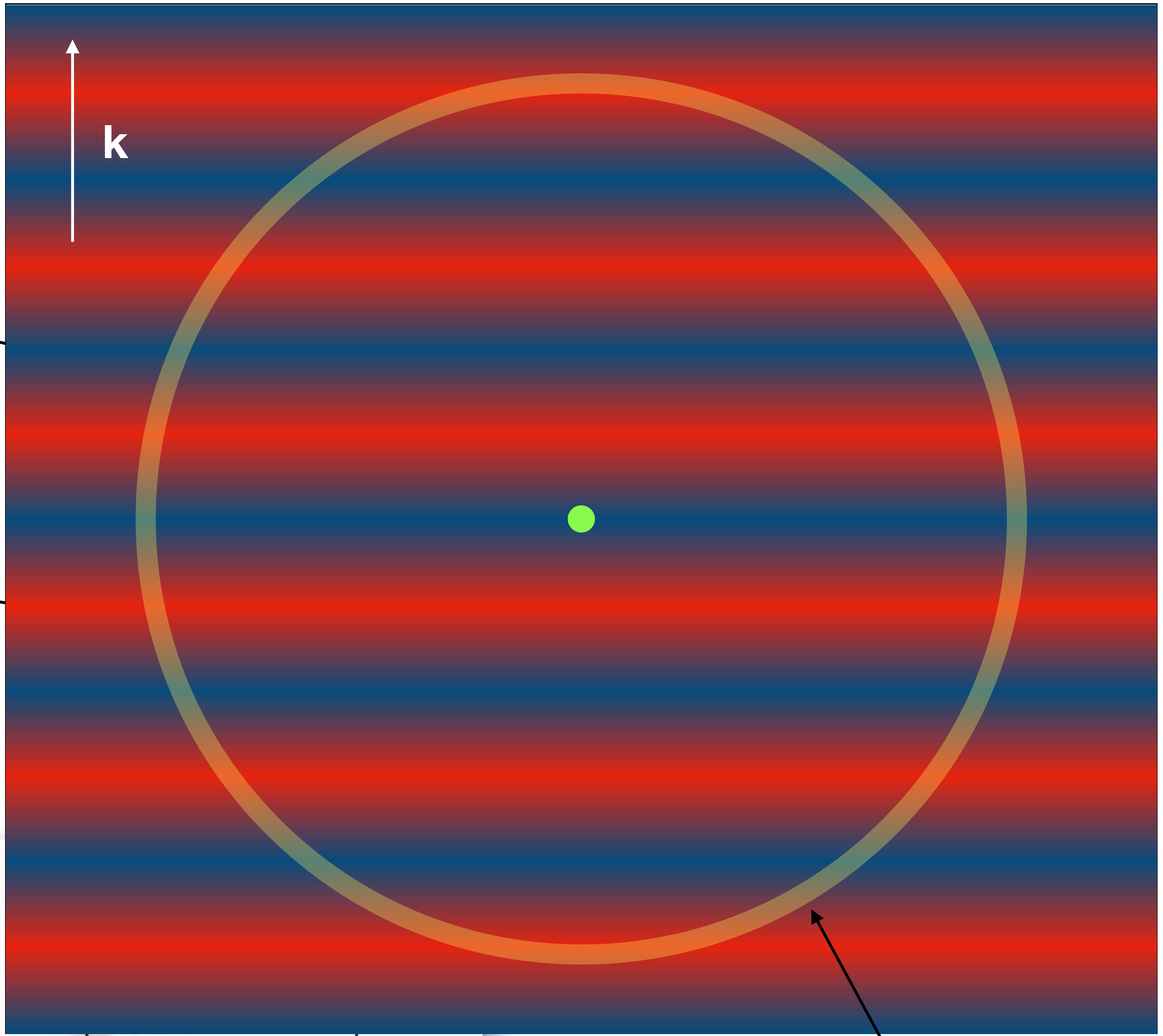


every point is in a local thermal equilibrium (black body)

$$I_\nu = \frac{4\pi\hbar\nu^3/c^2}{\exp\{2\pi\hbar\nu/k_B T\} - 1}$$

T(**x**,t)

Plane-wave inhomogeneity



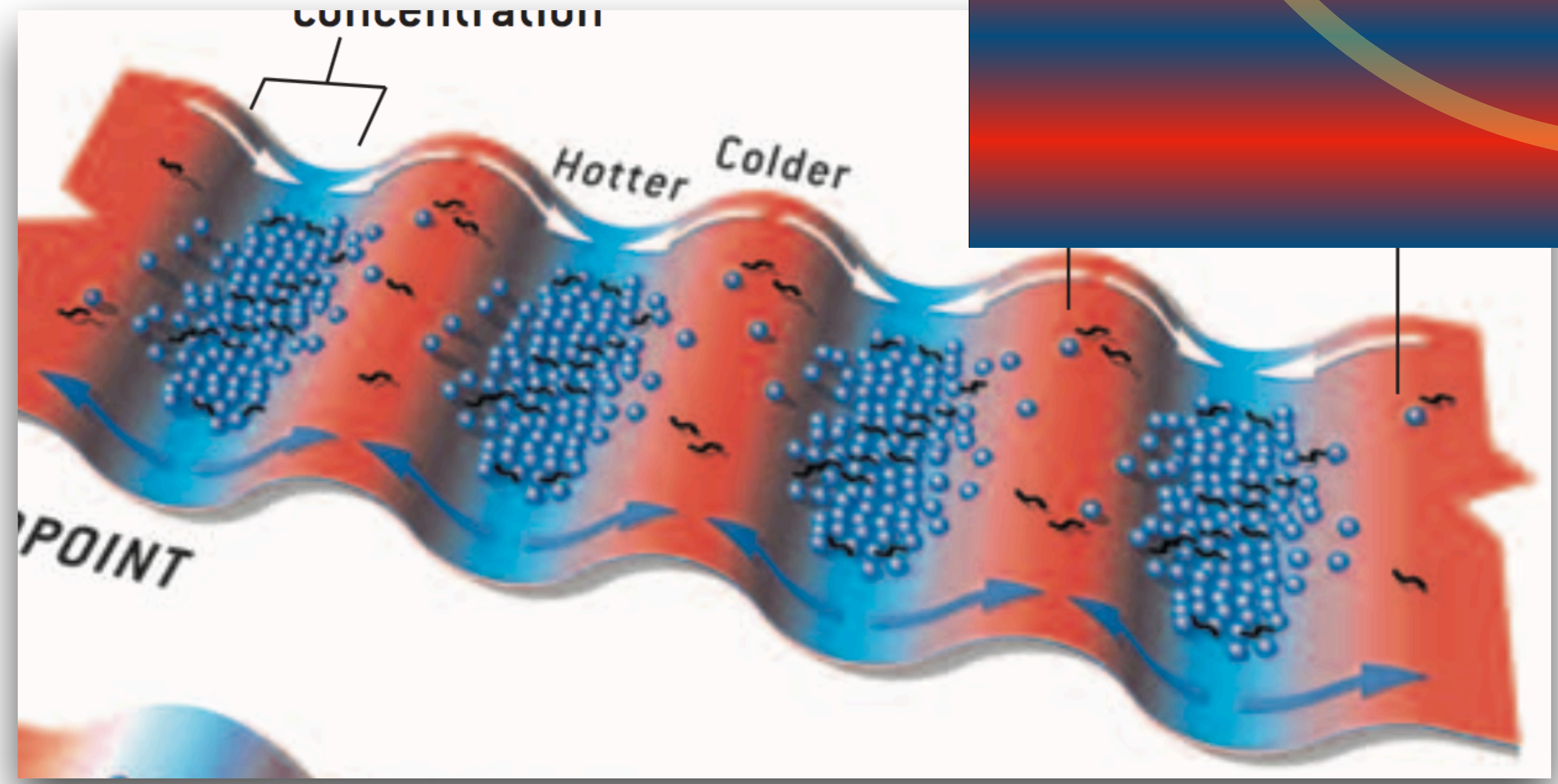
gravitational well

(hot regime)

gravitational wall

(cold regime)

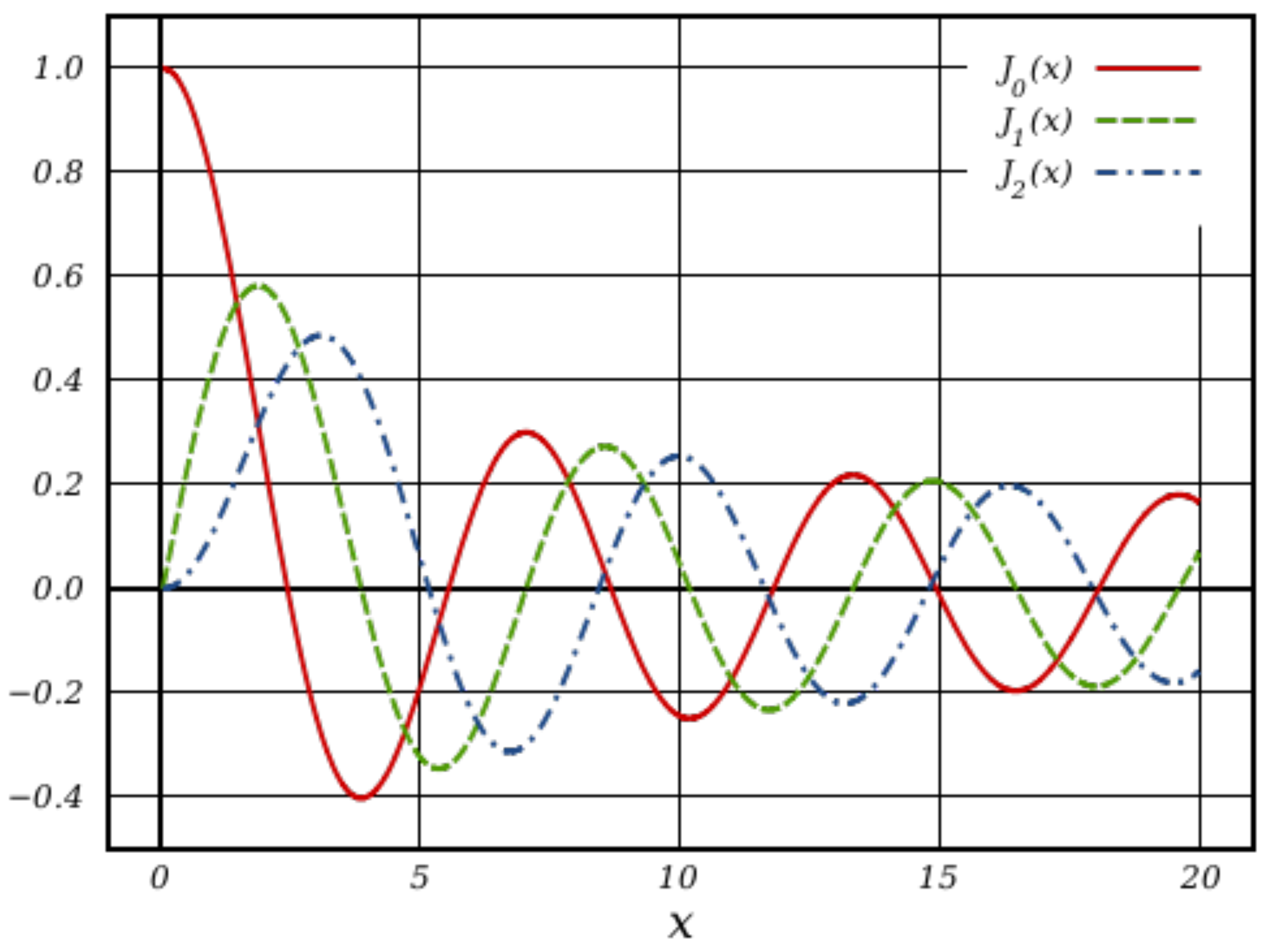
last scattering surface

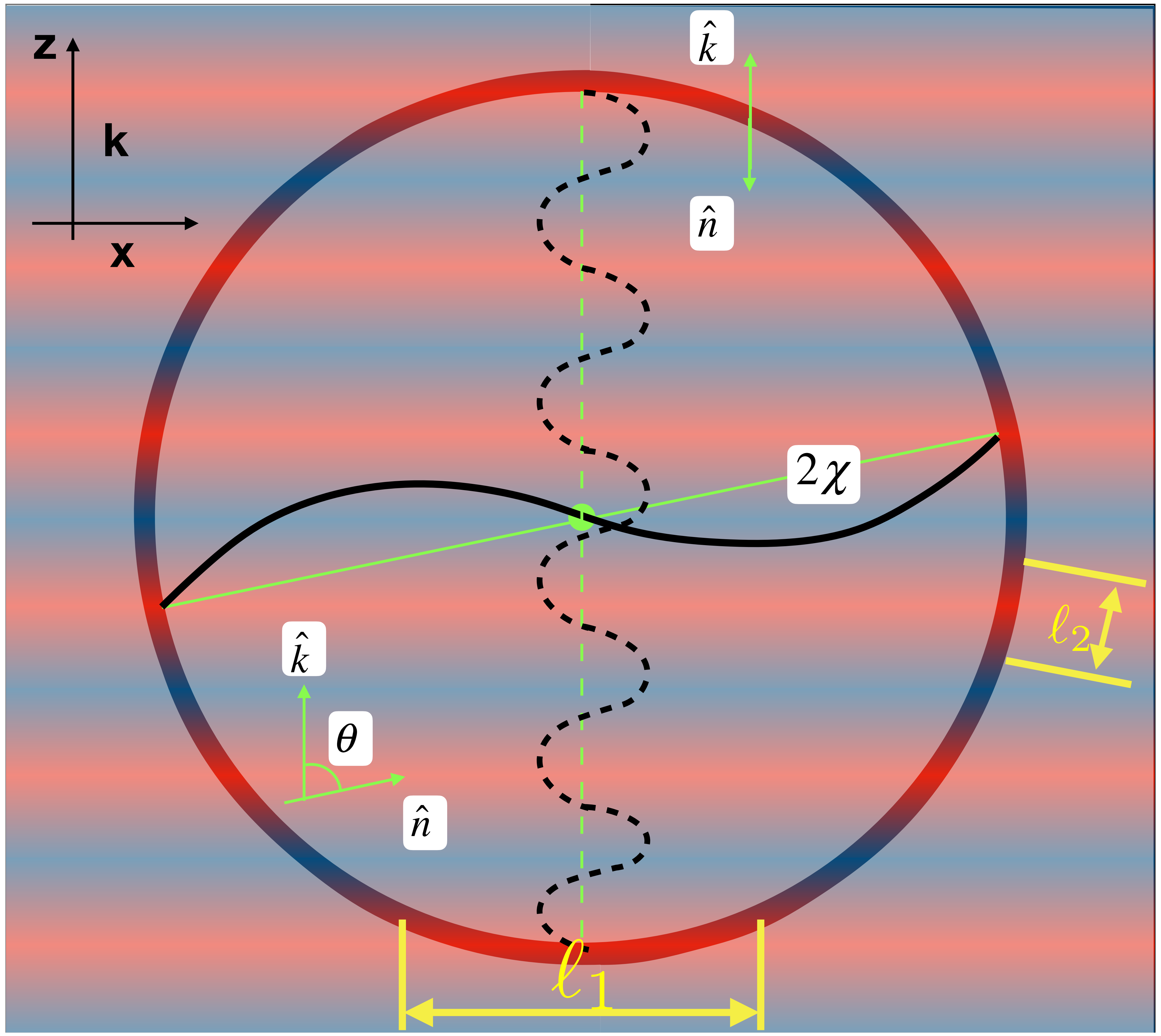


$$\psi(\hat{n}) = \psi(\chi,\vec{k})e^{i(\hat{k}\cdot\hat{n})k\chi}$$

$$= \psi(\chi,\vec{k})\sum_{\ell,m}[4\pi i^{\ell} Y^{*}_{\ell m}(\hat{k}) j_{\ell}(k\chi)] Y_{\ell m}(\hat{n})$$

$$= \psi(\chi,\vec{k})\sum_{\ell,m}\psi_{\ell m}(\chi,\vec{k}) Y_{\ell m}(\hat{n})$$





$$e^{i(\hat{k}\cdot\hat{n})k\chi} = 4\pi\sum_{\ell=0}\sqrt{\frac{2\ell+1}{4\pi}}\cdot i^{\ell}\cdot j_{\ell}(k\chi)Y_{\ell 0}(\hat{n}) \qquad (\hat{z}\,||\,\hat{k})$$

We use Spherical Harmonics and Spherical Bessel functions to expand the plane-wave

A plane wave can be expressed into a series of spherical wave

spatial inhomogeneity => angular anisotropy

[Hu & Dodelson
Annu. Rev. Astron. and Astrophys. 2002]

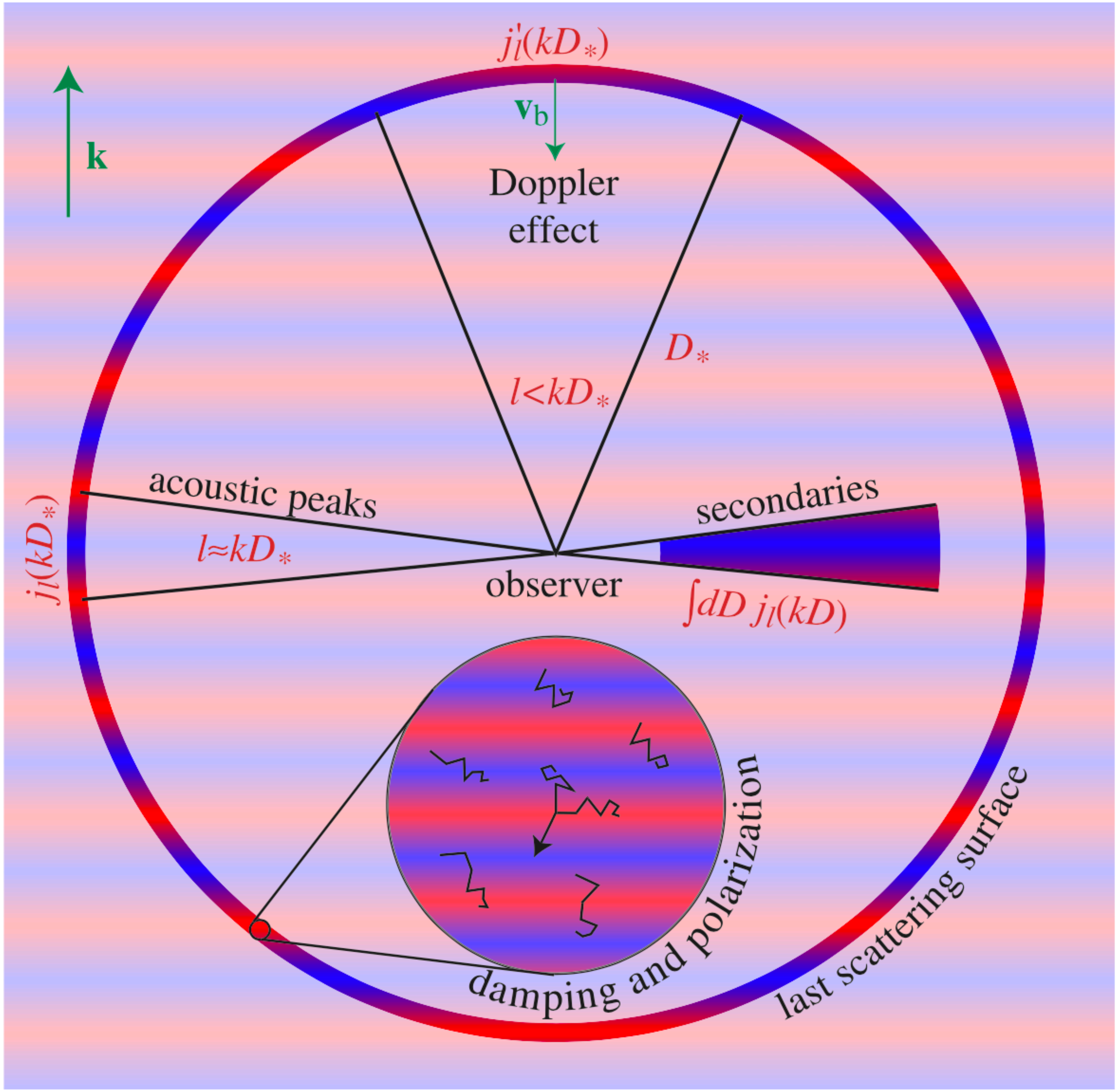
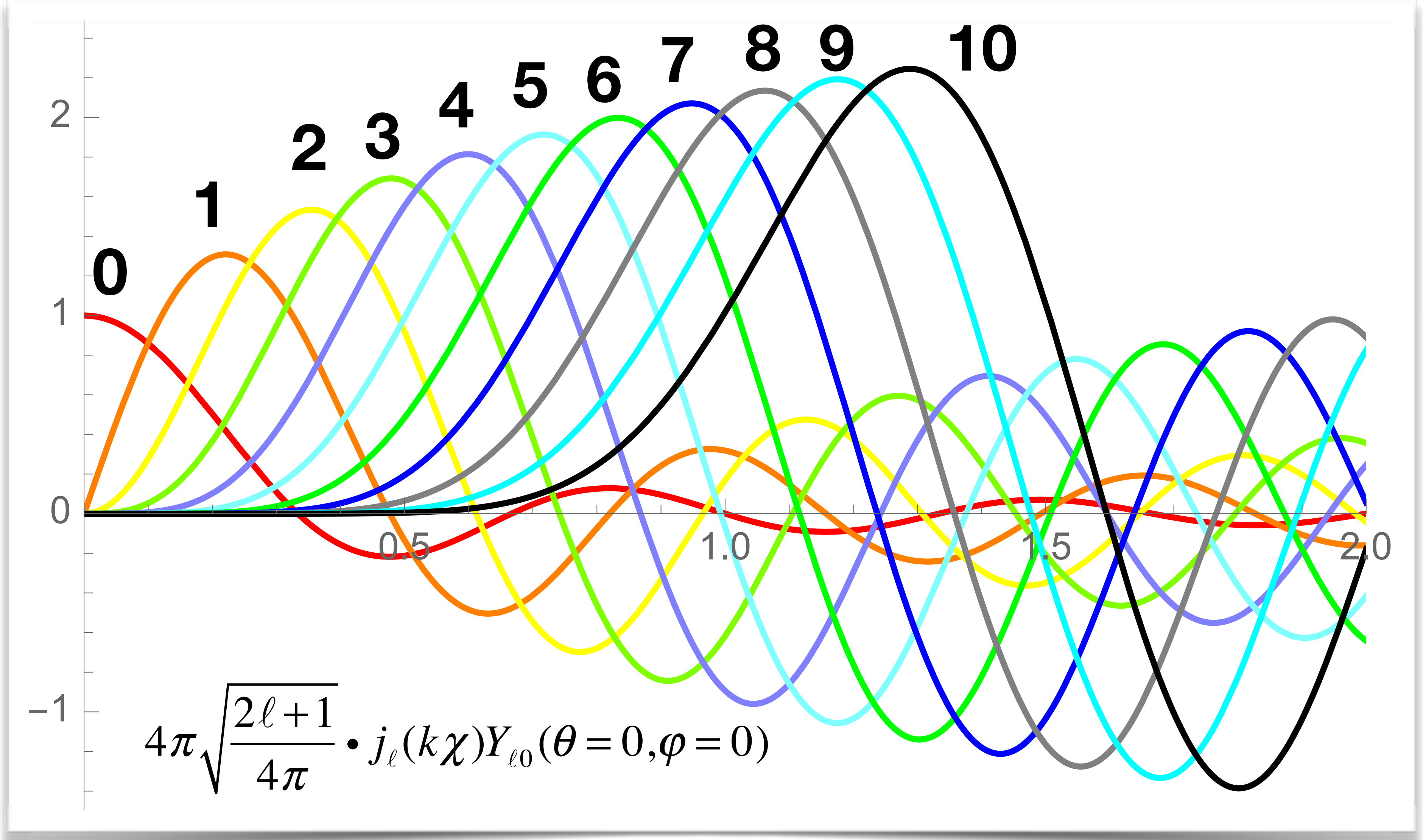
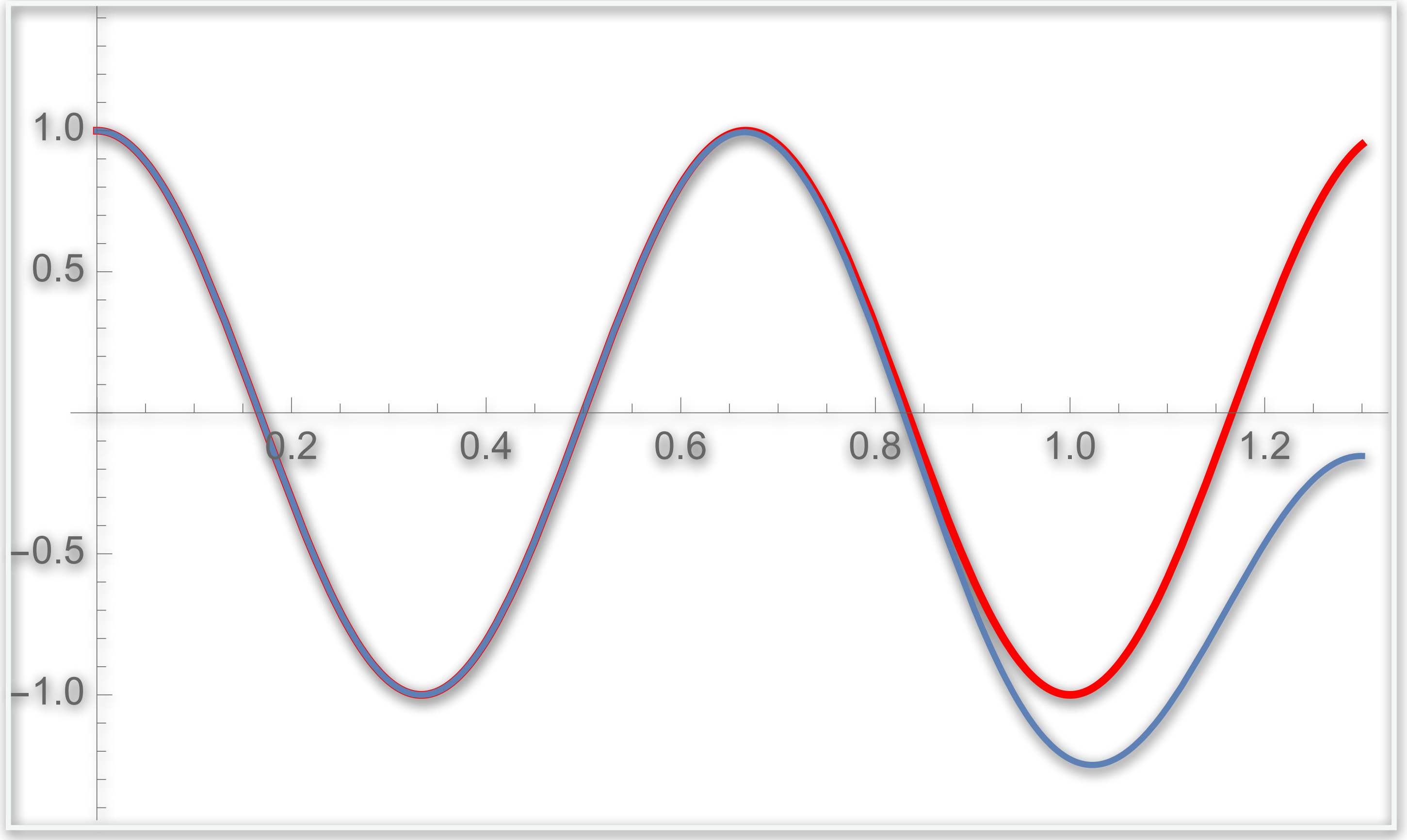


Plate 3: Integral approach. CMB anisotropies can be thought of as the line-of-sight projection of various sources of plane wave temperature and polarization fluctuations: the acoustic effective temperature and velocity or Doppler effect (see §3.8), the quadrupole sources of polarization (see §3.7) and secondary sources (see §4.2, §4.3). Secondary contributions differ in that the region over which they contribute is thick compared with the last scattering surface at recombination and the typical wavelength of a perturbation.

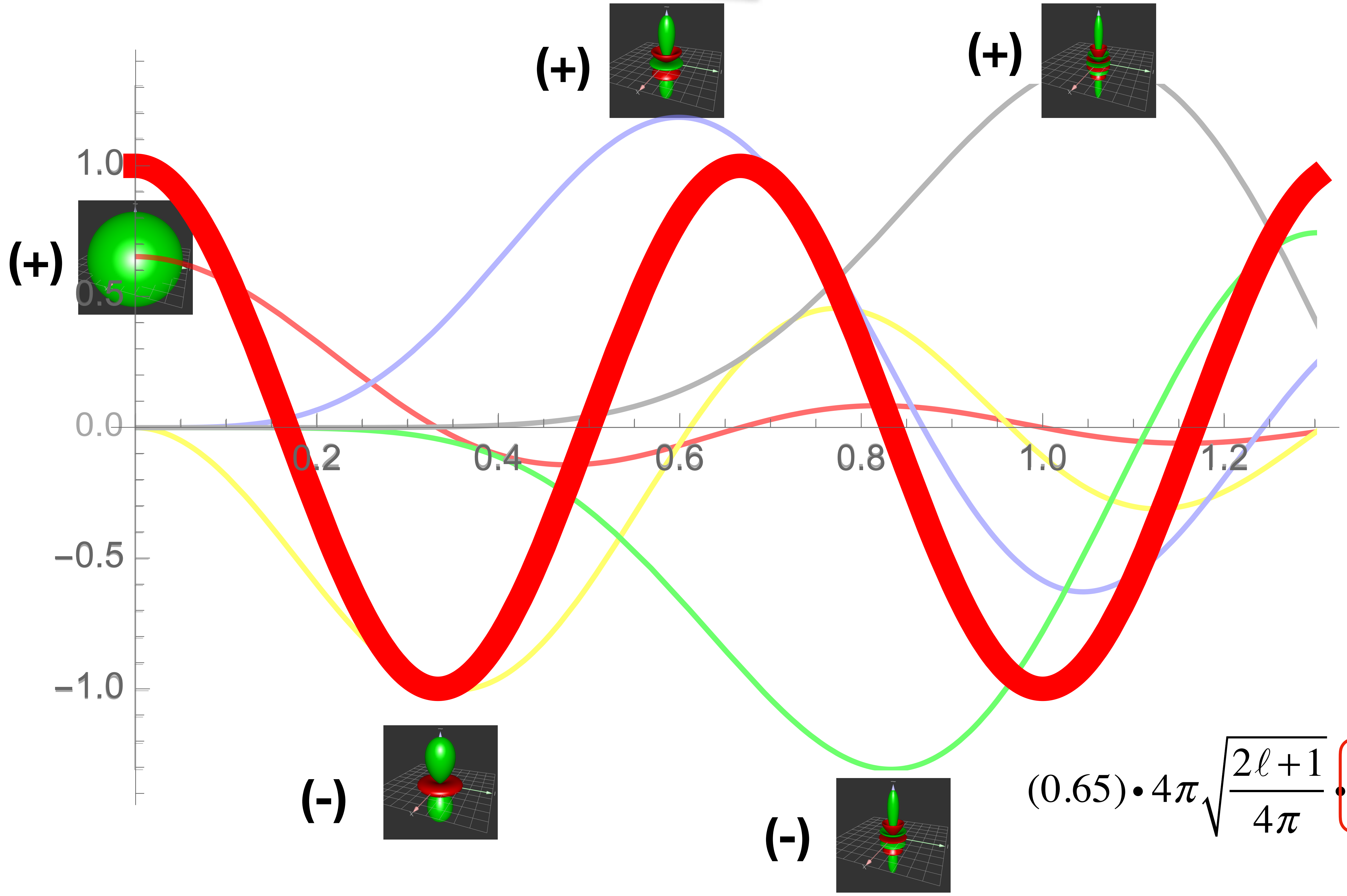




[Pb.]

plot the imaginary part

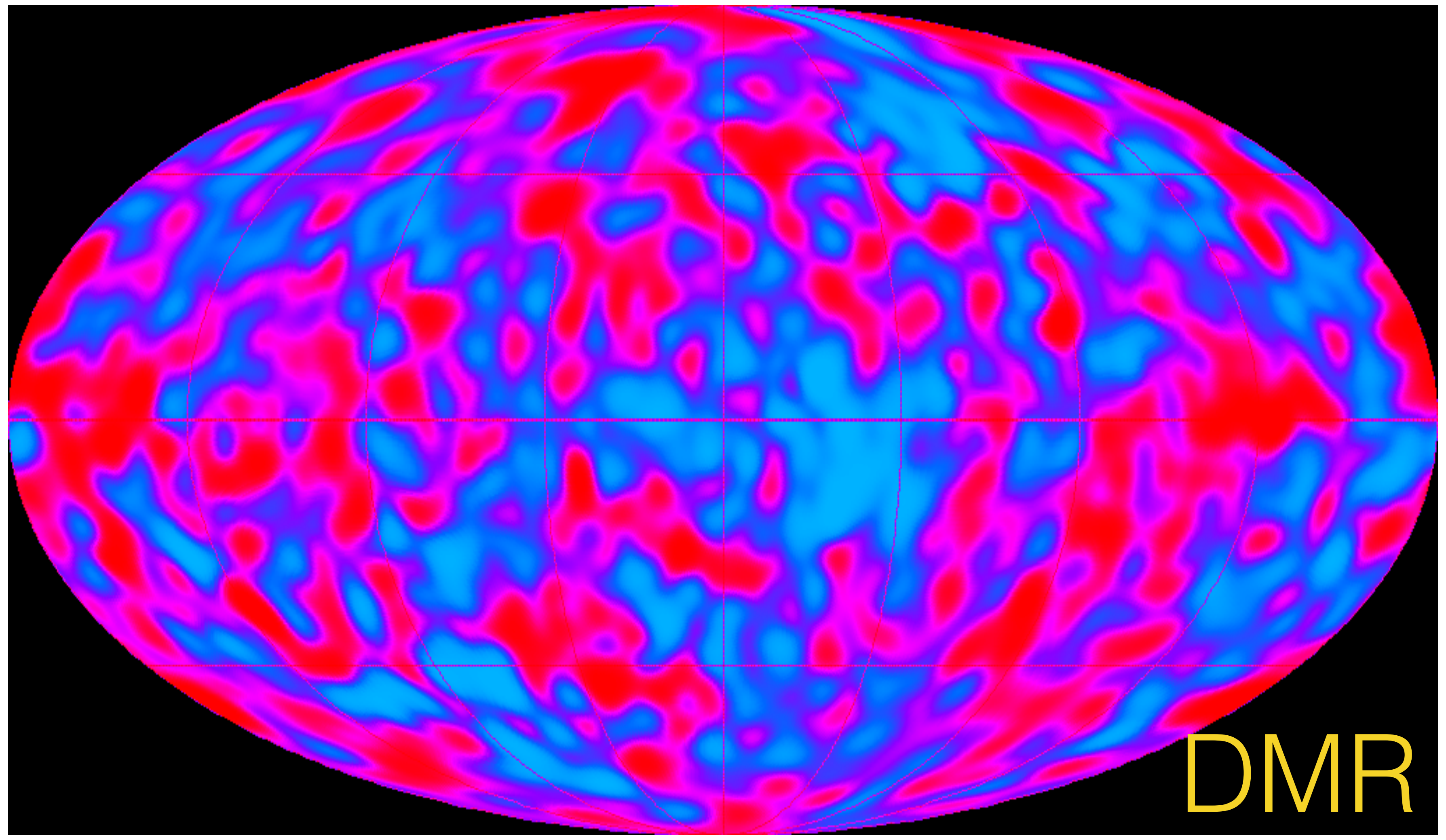
$e^{i(\hat{k}\bullet\hat{n})k\chi}$

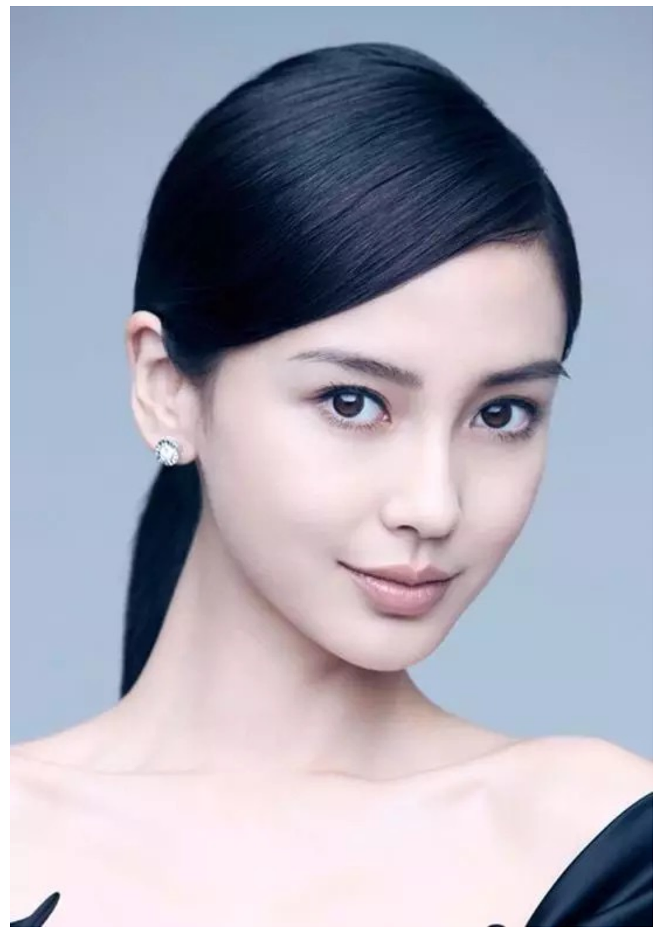


$$(0.65)\bullet 4\pi\sqrt{\frac{2\ell+1}{4\pi}}\bullet i^\ell \bullet j_\ell(k\chi)Y_{\ell 0}(\theta=0,\varphi=0)$$

1992年，COBE卫星（NASA）

DMR——测量各向异性的微分测量仪





陆埮老师：“她是我们能够用光学手段看到的宇宙自诞生之日起的第一张baby face”

在红移1100之前(宇宙诞生38万年之前)，宇宙的物质状态为“一锅等离子体热汤”，各种物质组分紧紧地耦合在一起，其中最主要的是自由电子和光子的Thompson散射(弹性散射)

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q'}) + \gamma(\vec{p'})$$

该过程在红移1100之前，频繁发生无数次!
从而使得，“这锅等离子体热汤”达到热平衡。

当“这锅热汤”的温度降到大约3000K（约0.1eV）时，
电子动能（系统热能），不足以抵抗氢原子的第一电离能（13.6eV），电子－质子形成中性氢原子。
该过程几乎瞬时完成，之后就几乎没有自由电子

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q}') + \gamma(\vec{p}')$$

之后，光子几乎自由地传播至现在！
(free streaming)

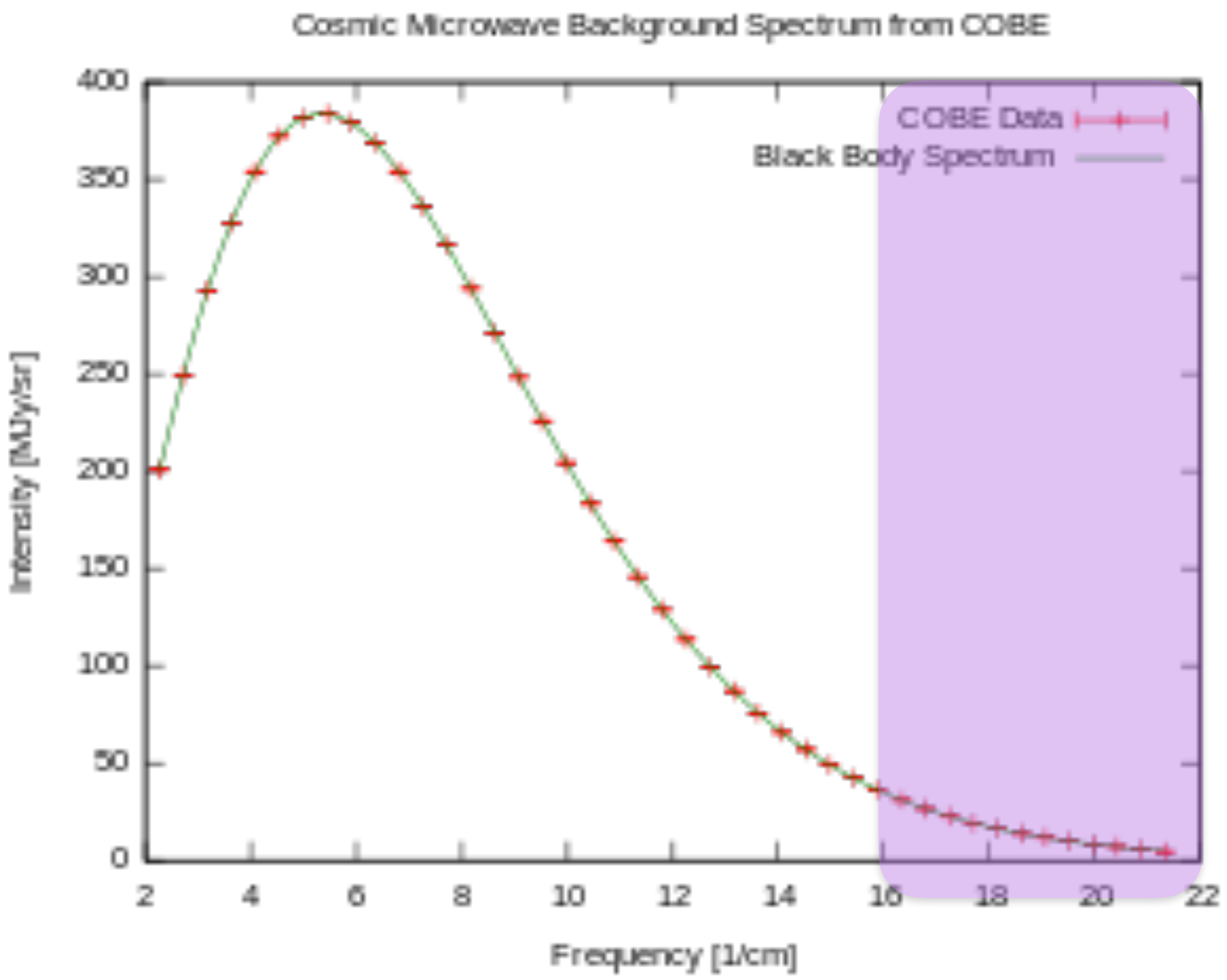
既然氢原子的第一电离能是13.6eV，

为什么Thompson散射过程不在

宇宙温度降低到30万K时就停止呢?

宇宙中的，重子（电子）/ 光子比，非常非常低！ $\eta \sim 10^{-10}$

一个电子周围包裹着一群光子，这些光子数目按照黑体谱分布



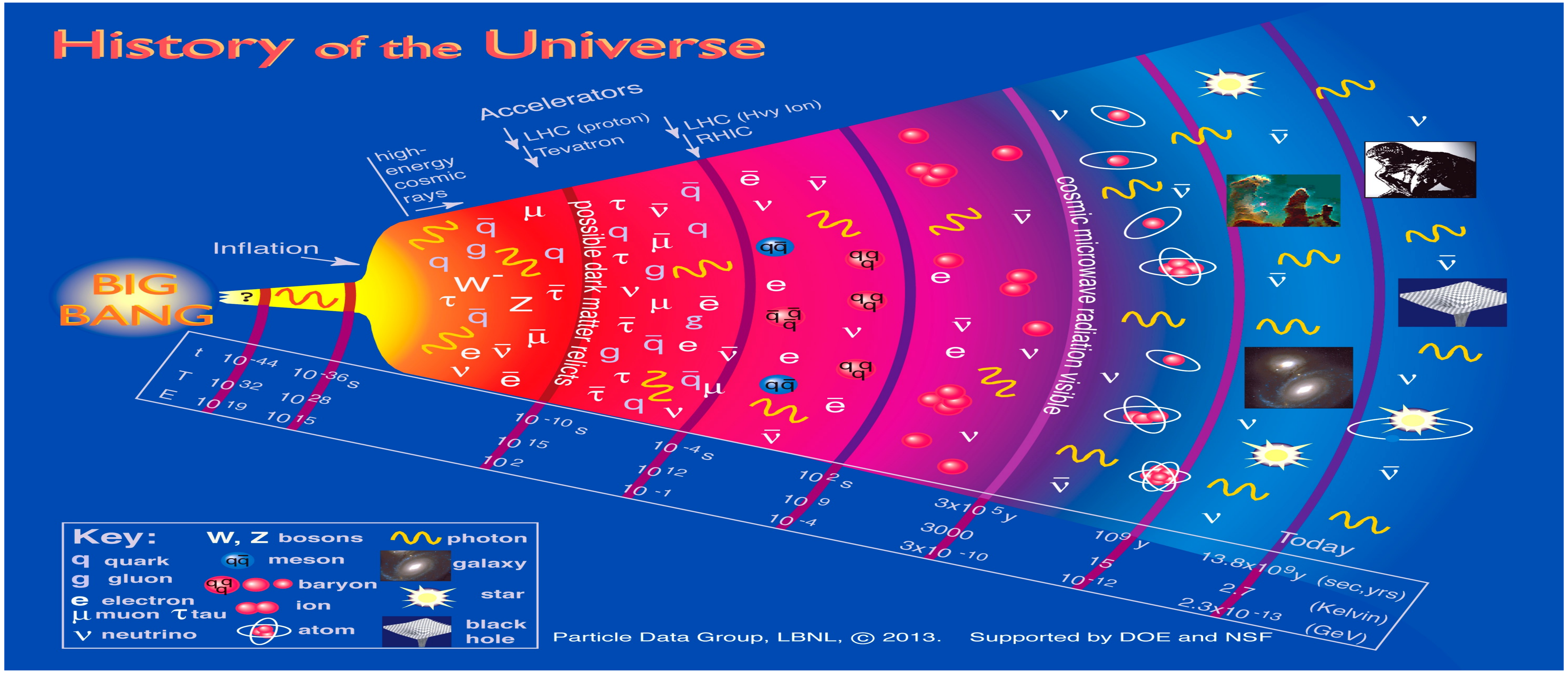
一个电子只需要一个光子
就能完成这个散射过程

高能光子比重小，
但是整体数目
相比于电子并不少

发生的能标比13.6eV要远低！

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q}') + \gamma(\vec{p}')$$

在红移1100时，该过程最后一次发生，
之后自由电子和自由质子迅速形成中性氢。
因此，在时空图上，这个过程可以看作是，
薄薄的一层（Last Scattering Surface）



上述过程的数学描述

1. 声学震荡 / acoustic oscillation

光子温度扰动

$\Theta = \Delta T/T$

$$m_{\rm eff}\ddot{\Theta} + k^2c^2\Theta/3 \simeq m_{\rm eff}\, g$$

(受迫谐振子)

$m_{\rm eff} = 1 + R$

重子密度比

$$g = -k^2c^2\Psi/3 - \ddot{\Phi}$$

(静态势阱近似)

$\tilde{\Theta} = \Theta + \Psi$

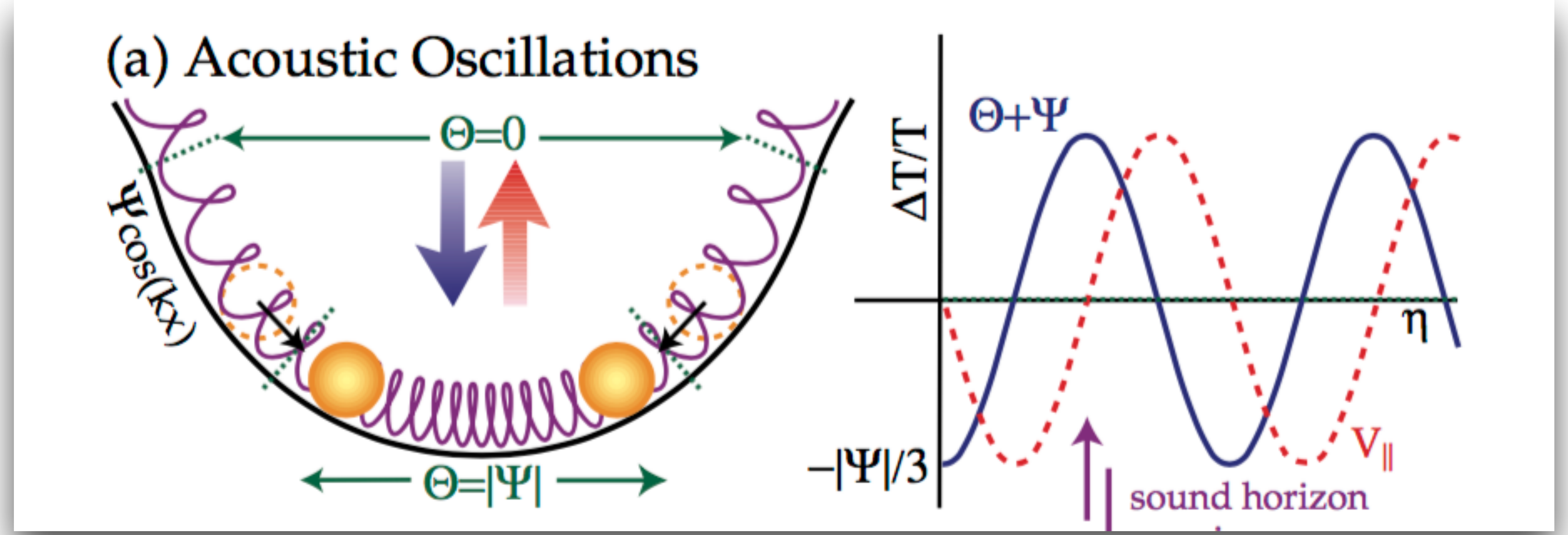
$$\ddot{\tilde{\Theta}} + k^2c^2\tilde{\Theta}/3 \simeq 0$$

绝热初始条件:

$\Theta(0) = -2\Psi/3, \dot{\Theta}(0) = 0,$

$$\tilde{\Theta} = \Psi\cos(ks)/3$$

Hu et.al. 97’

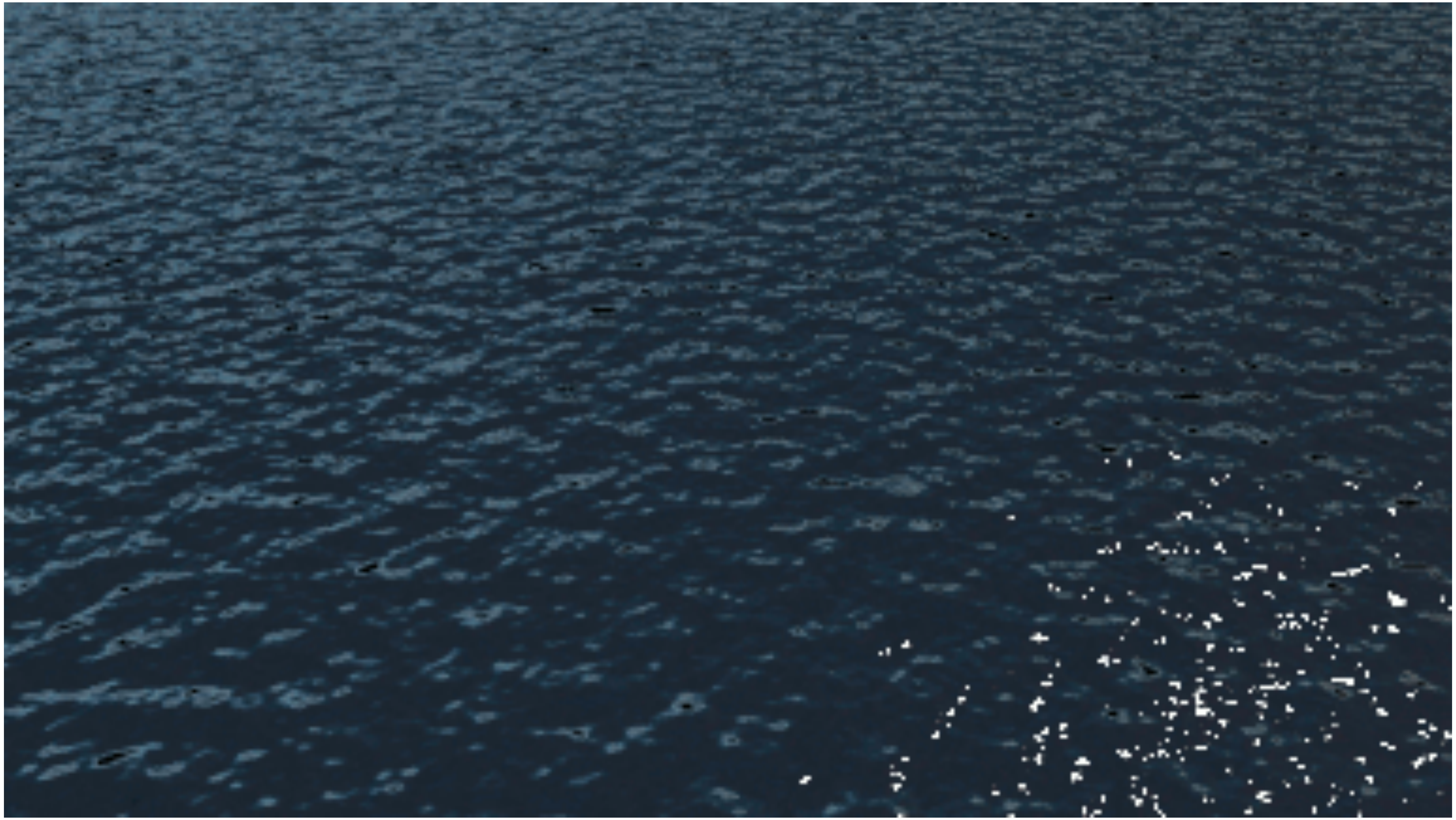


重子－光子等离子体会塌缩到引力势阱中

压缩到等离子体声学视界半径之下，
光压就会阻止引力继续塌缩，从而形成声学震荡

就这样一直震下去。。。

before recombination 波光粼粼

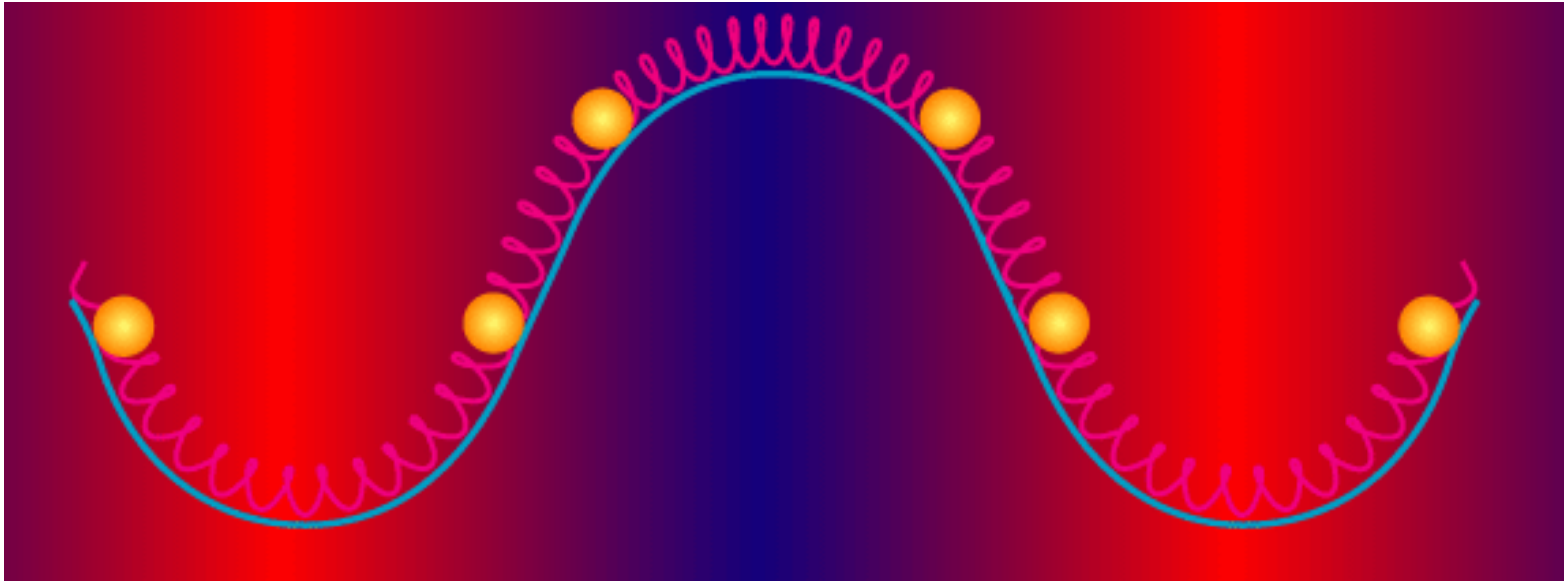


Winter is coming ...

直到，形成中性氢

光子，不再受重子拖曳，从引力势阱中逃逸出去
其动能转化为引力势能，光子能量损失 − | \Psi |

Compressing a gas heats it up. Letting it expand cools it down.
The CMB is locally hotter in regions where the acoustic wave causes compression and cooler where it causes rarefaction:



[credit: W. Hu]

[credit: W. Hu]

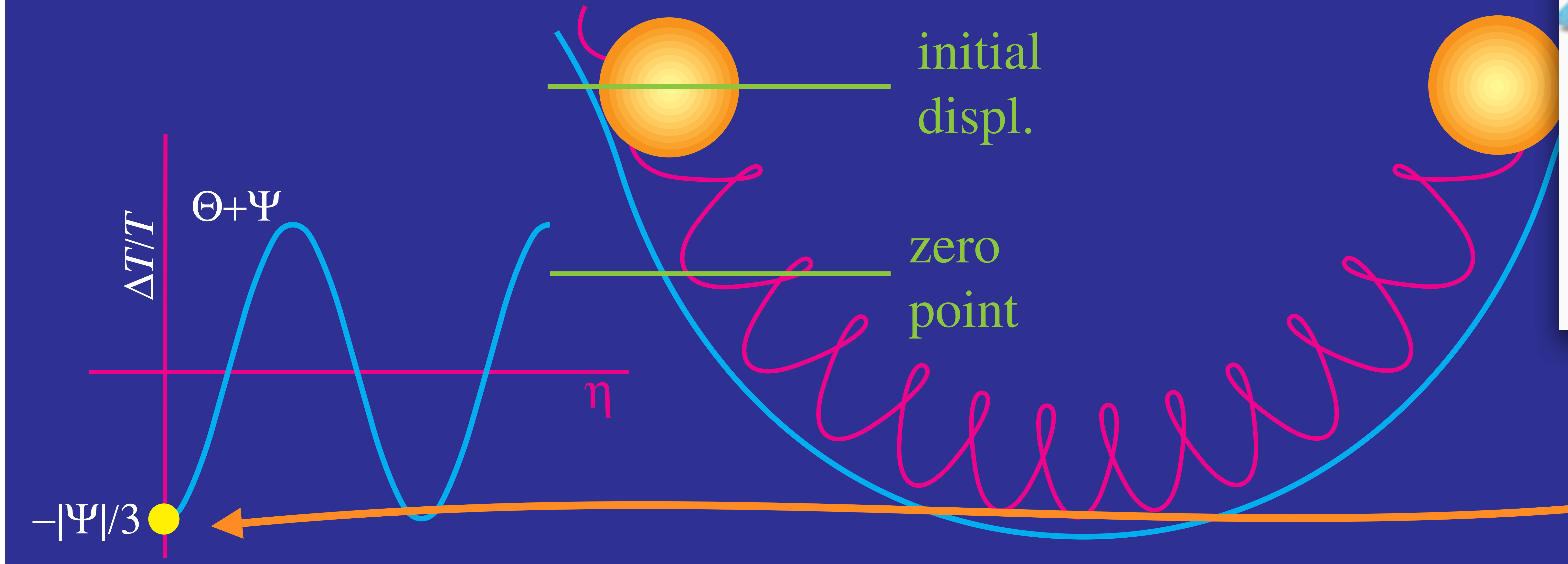
Acoustic Oscillations

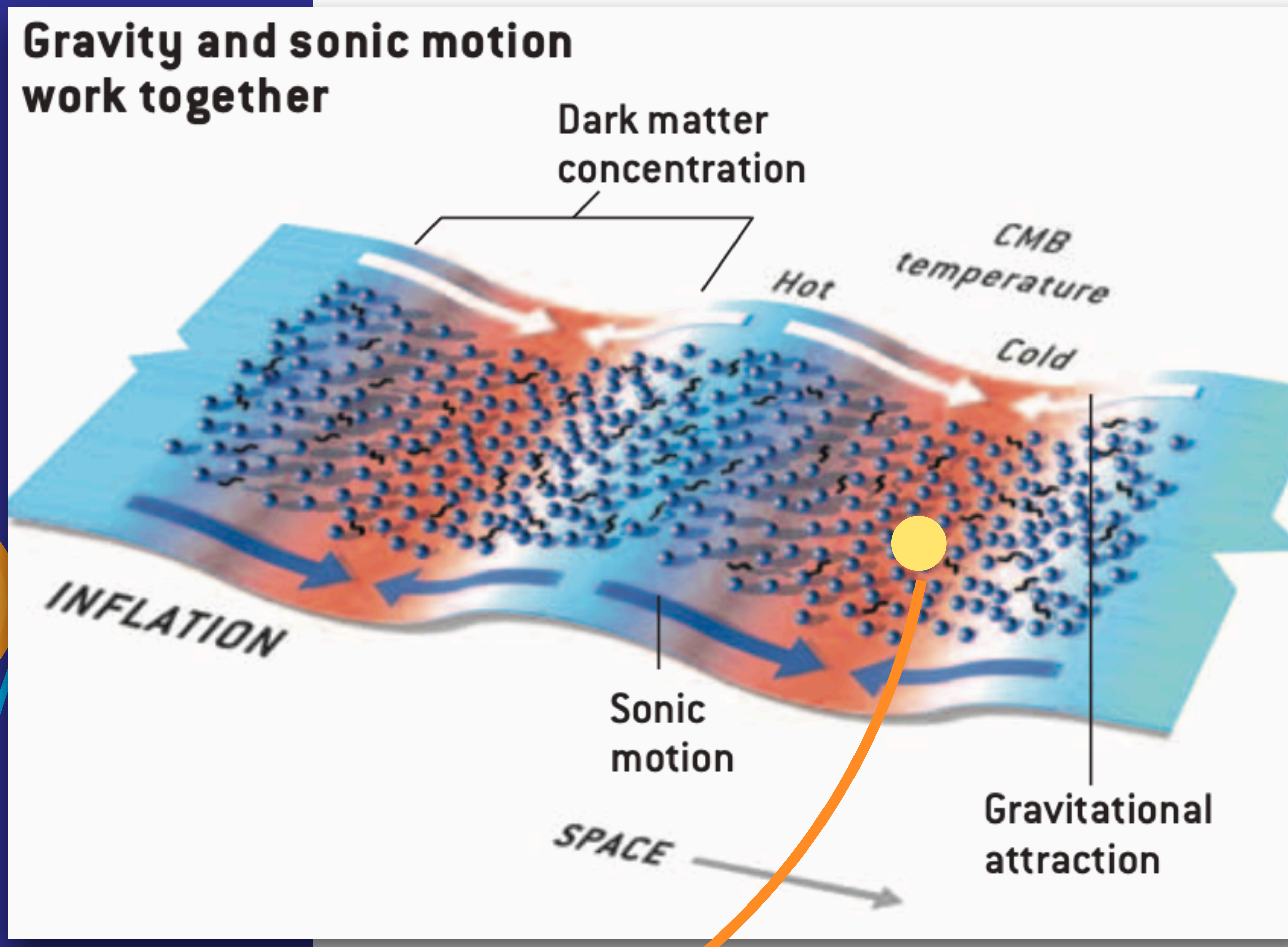
- Photon pressure resists compression in potential wells
- Acoustic oscillations
- Gravity displaces zero point
  $\Theta \equiv \delta T/T = -\Psi$
- Oscillation amplitude = initial displacement from zero pt.
  $\Theta - (-\Psi) = 1/3\Psi$

a fix spatial spot

@Initial time

A cold spot





Peebles & Yu (1970)

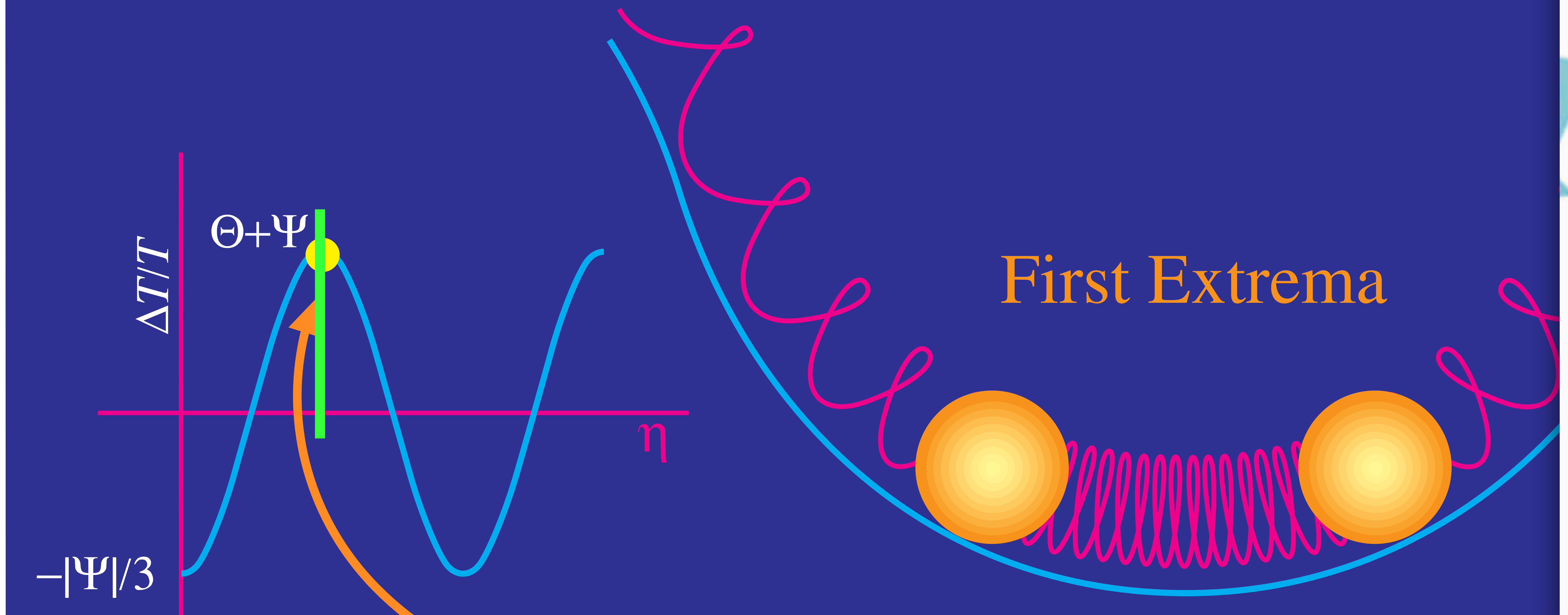
Hu & Sugyama (1995)

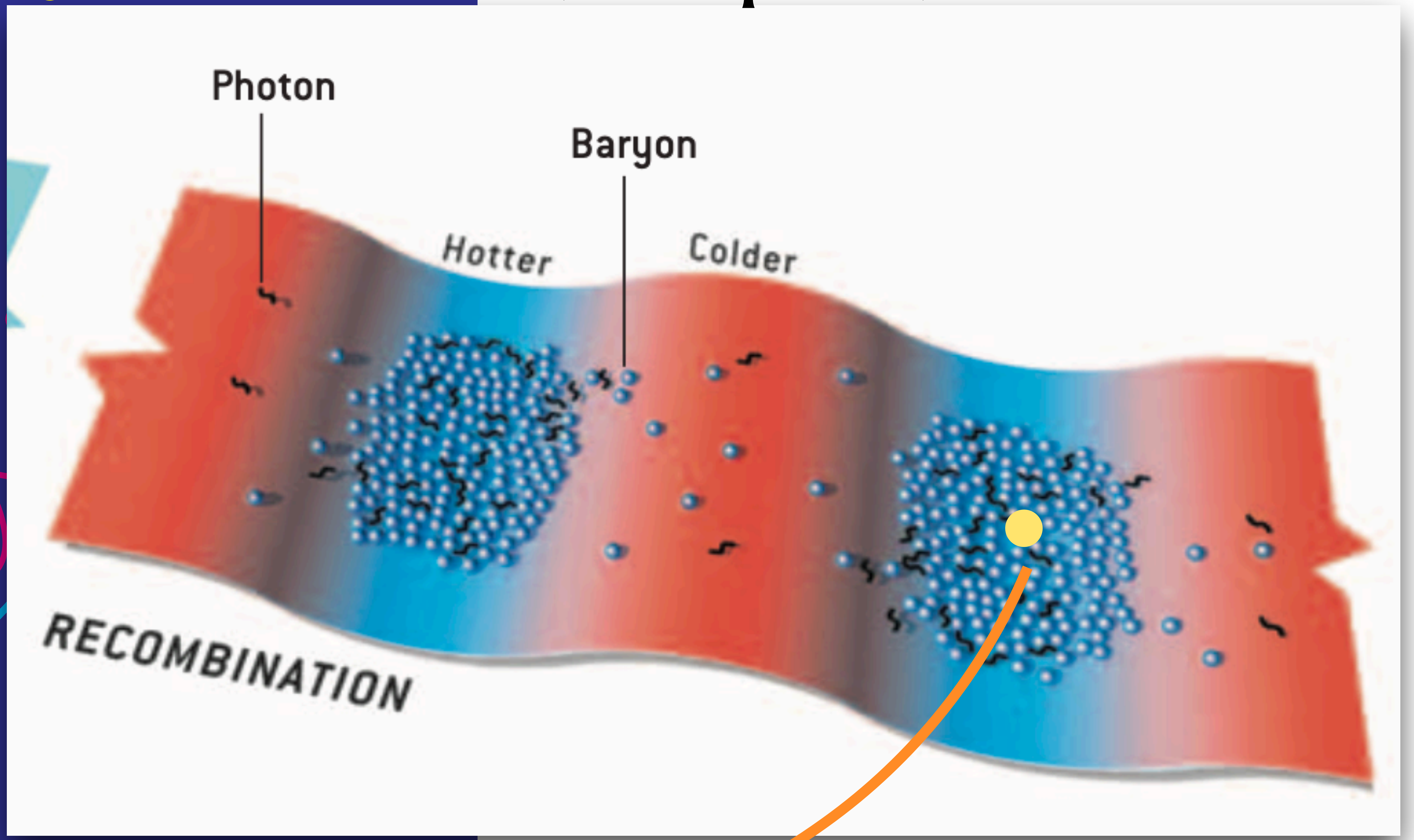
[credit: W. Hu]

Acoustic Oscillations

- Photon pressure resists compression in potential wells
- Acoustic oscillations
- Gravity displaces zero point $\Theta \equiv \delta T/T = -\Psi$
- Oscillation amplitude = initial displacement from zero pt. $\Theta - (-\Psi) = 1/3\Psi$
- Gravitational redshift: observed $(\delta T/T)_{obs} = \Theta + \Psi$ oscillates around zero

the same spatial spot @ recom time becomes hot spot due to gravity compression (first peak)





Peebles & Yu (1970)

Hu & Sugyama (1995)

[credit: W. Hu]

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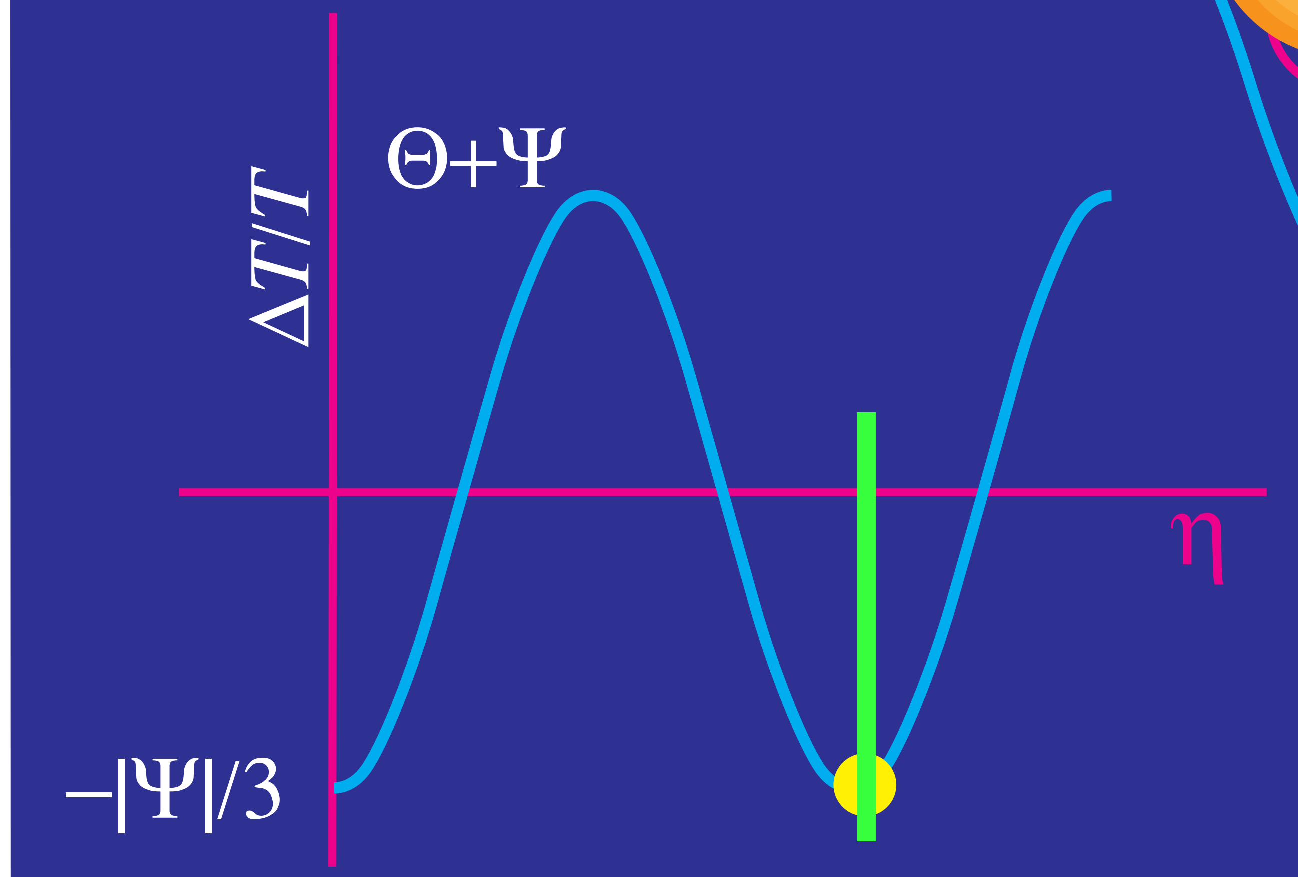

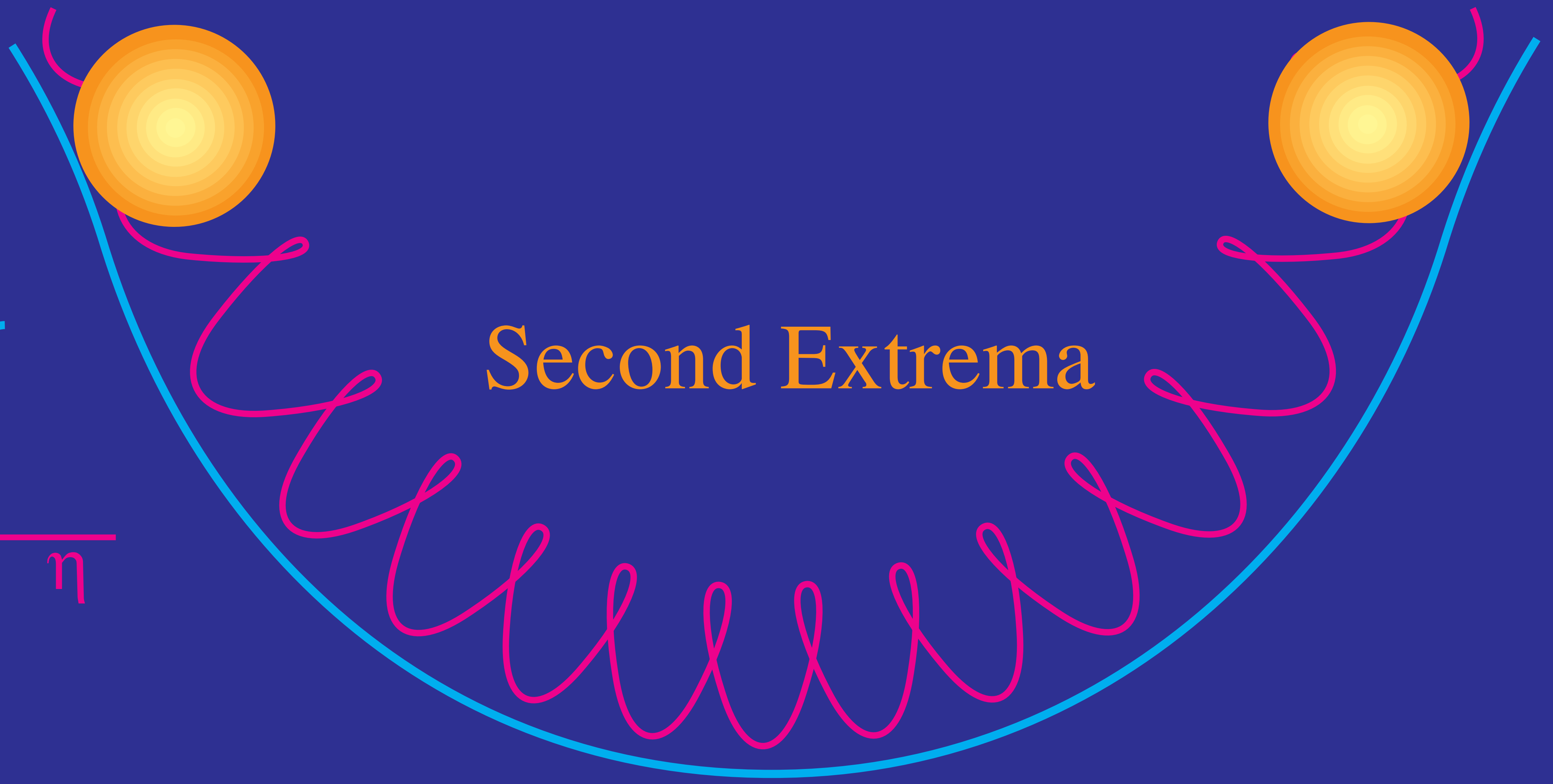

Peebles & Yu (1970)

Hu & Sugyama (1995)

Now, consider another spatial spot, which located at a smaller well. From initial time, till recom time, it oscillate 2pi. (with higher frequency)

initially, it is a cold spot

in the middle time between initial and recom time, it becomes a hot spot

[credit: W. Hu]

Acoustic Oscillations

- Photon pressure resists compression in potential wells
- Acoustic oscillations
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  $\Theta \equiv \delta T/T = -\Psi$
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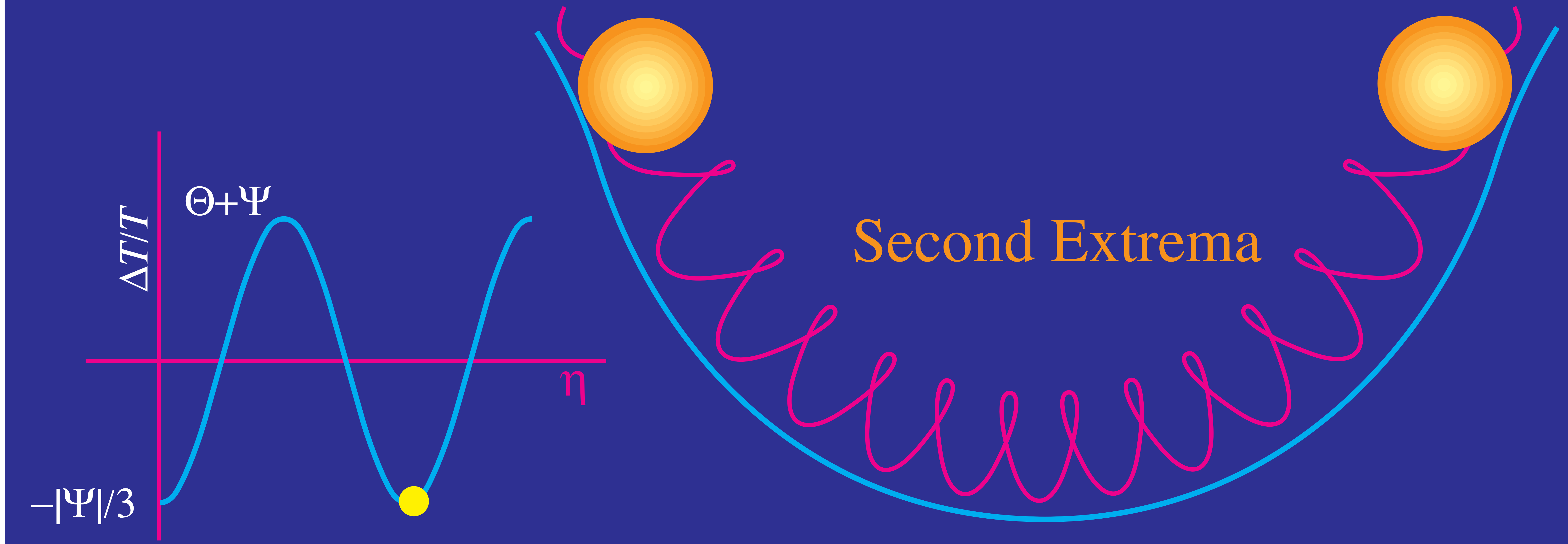
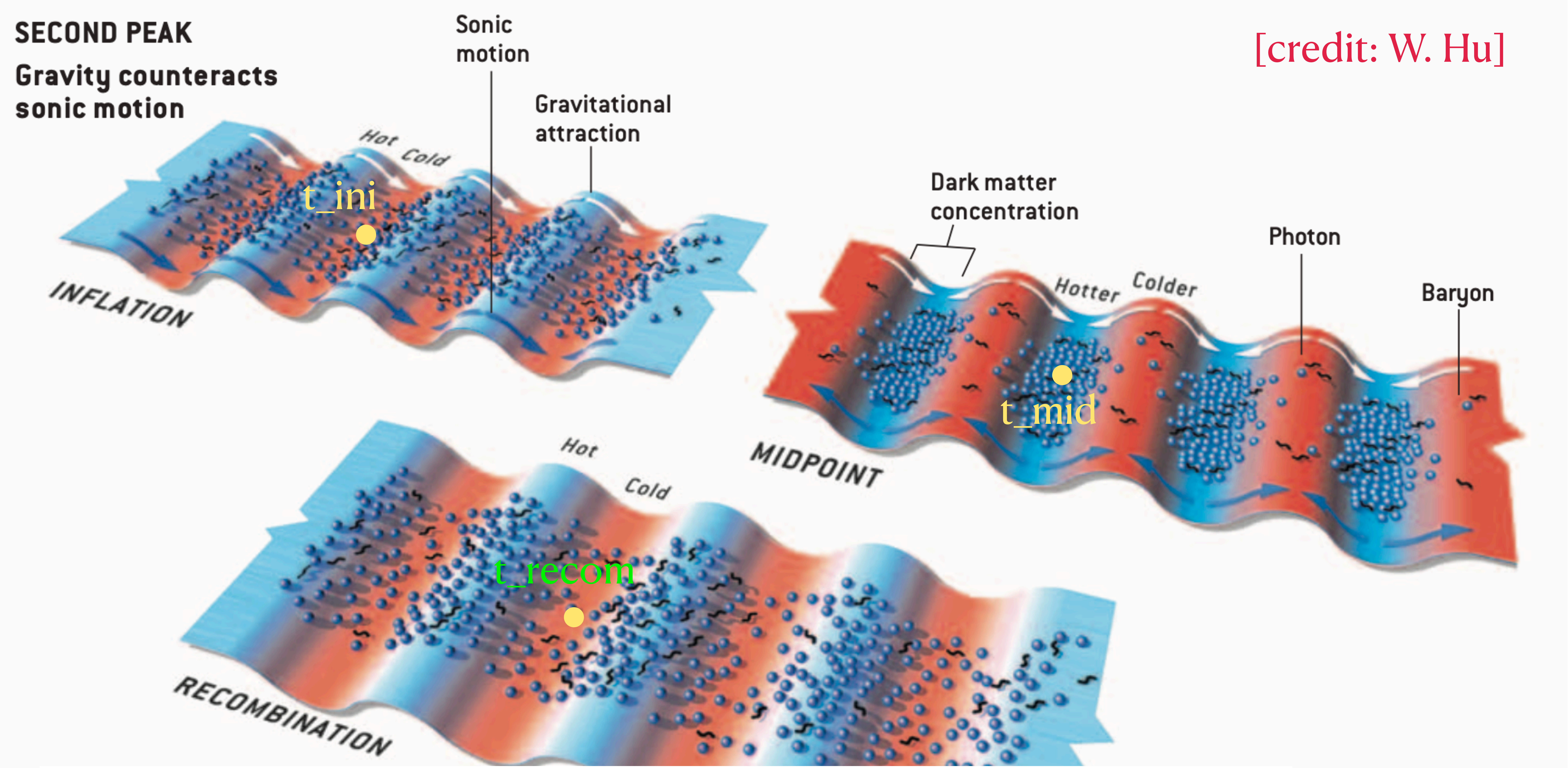
  $\Theta - (-\Psi) = 1/3\Psi$
- Gravitational redshift: observed
  $(\delta T/T)_{obs} = \Theta + \Psi$
  oscillates around zero



Peebles & Yu (1970)

Hu & Sugyama (1995)

Finally, at recom time, it becomes back to a cold spot. (second peak)

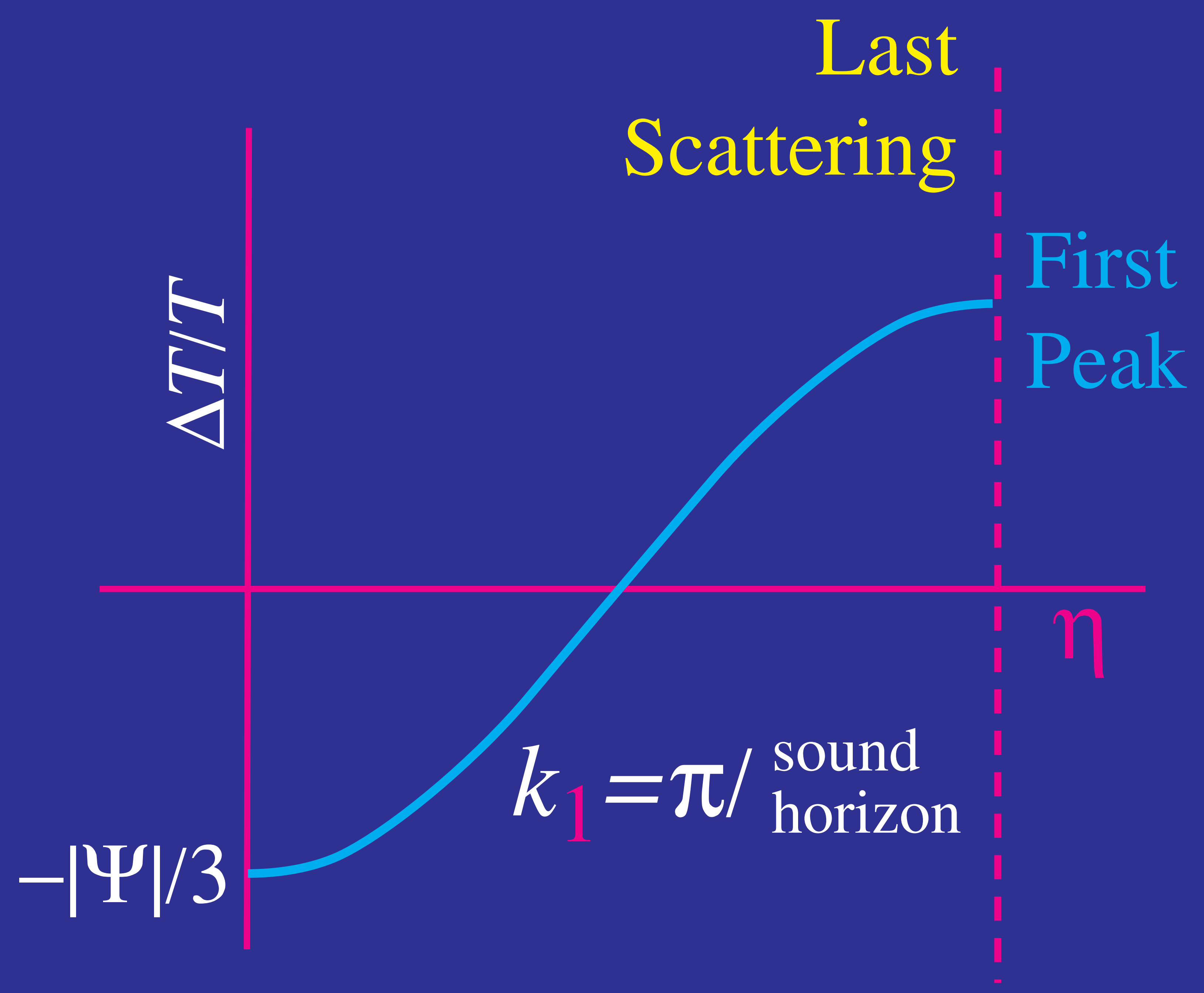


After this midpoint, gas pressure pushes baryons and photons out of the troughs (*blue arrows*) while gravity tries to pull them back in (*white arrows*). This tug-of-war decreases the temperature differences, which explains why the second peak in the power spectrum is lower than the first.

[credit: W. Hu]

Harmonic Peaks

- Oscillations frozen at last scattering
- Wavenumbers at extrema = peaks
- Sound speed c_s



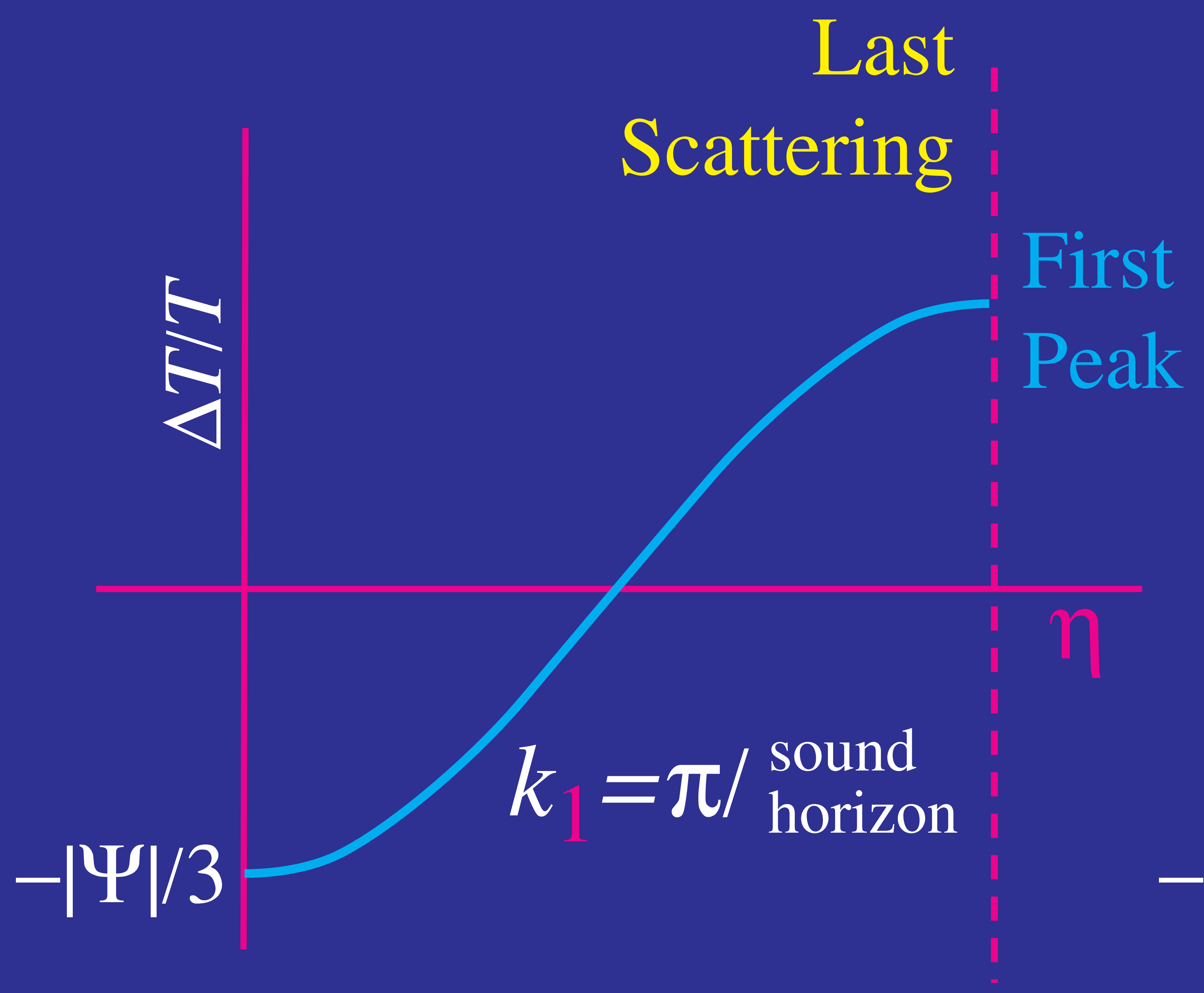
Doroshkevich, Zel'dovich & Sunyaev (1978); Bond & Efstathiou (1984); Hu & Sugiyama (1995)

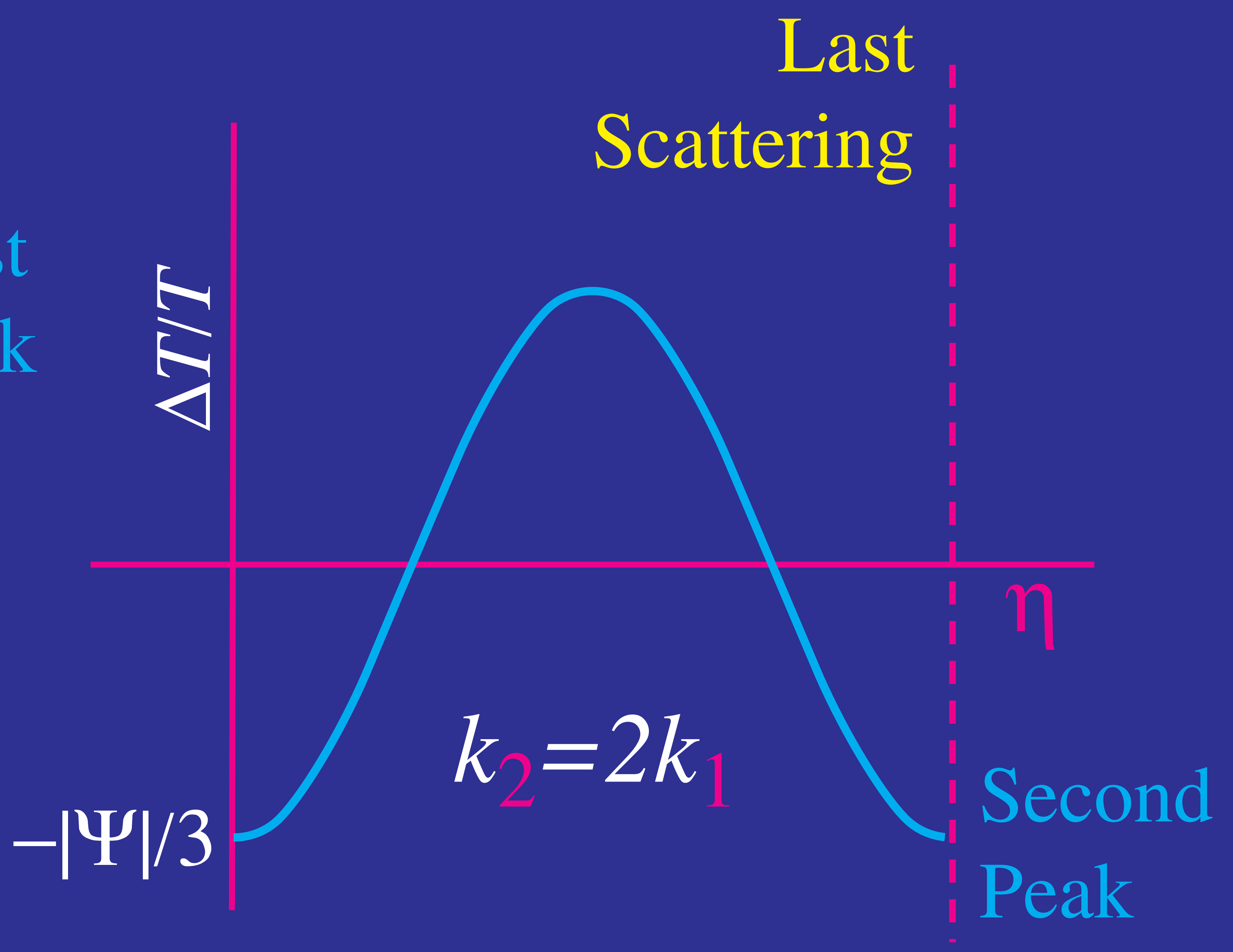
[credit: W. Hu]

Harmonic Peaks

- Oscillations frozen at last scattering
- Wavenumbers at extrema = peaks
- Sound speed c_s

- Frequency $\omega = kc_s$; conformal time η
- Phase $\propto k$; $\phi = \int_0^{\text{last scattering}} d\eta\, \omega = k$ sound horizon
- Harmonic series in sound horizon
  $\phi_n = n\pi \rightarrow k_n = n\pi /$ sound horizon





Doroshkevich, Zel'dovich & Sunyaev (1978); Bond & Efstathiou (1984); Hu & Sugiyama (1995)

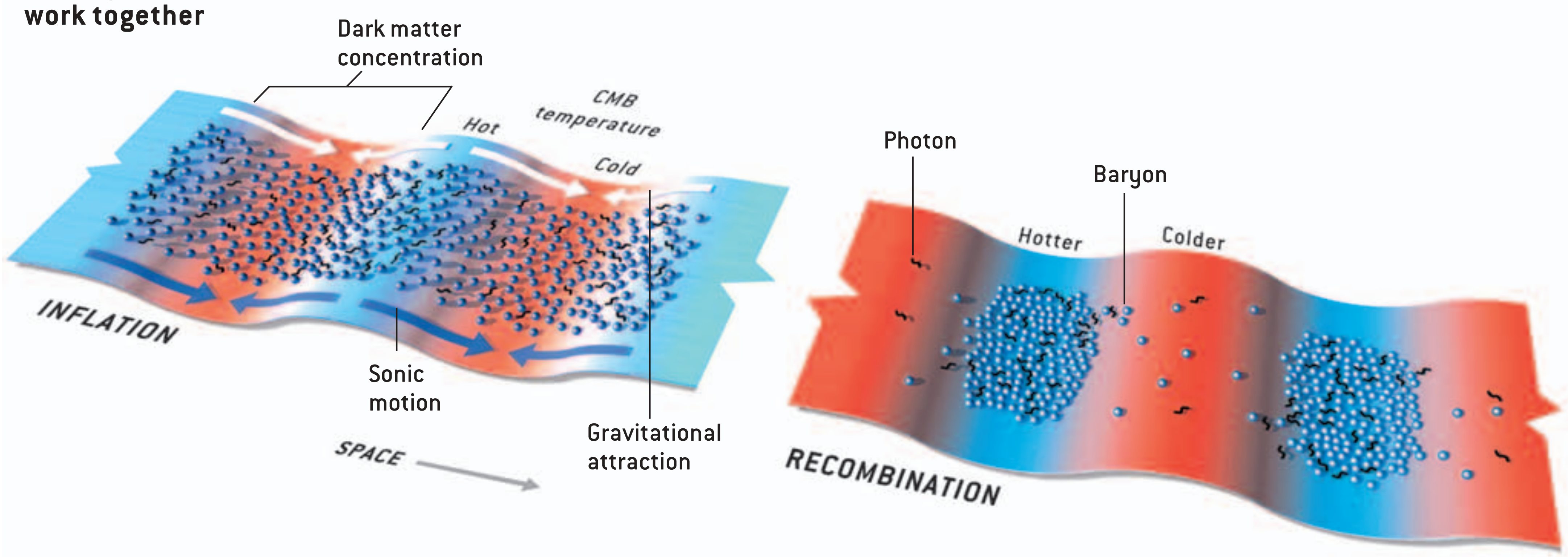
GRAVITATIONAL MODULATION

[credit: W. Hu]

INFLUENCE OF DARK MATTER modulates the acoustic signals in the CMB. After inflation, denser regions of dark matter that have the same scale as the fundamental wave (*represented as troughs in this potential-energy diagram*) pull in baryons and photons by gravitational attraction. (The troughs are shown in red because gravity also reduces the temperature of any escaping photons.) By the time of recombination, about 380,000 years later, gravity and sonic motion have worked together to raise the radiation temperature in the troughs (*blue*) and lower the temperature at the peaks (*red*).

FIRST PEAK

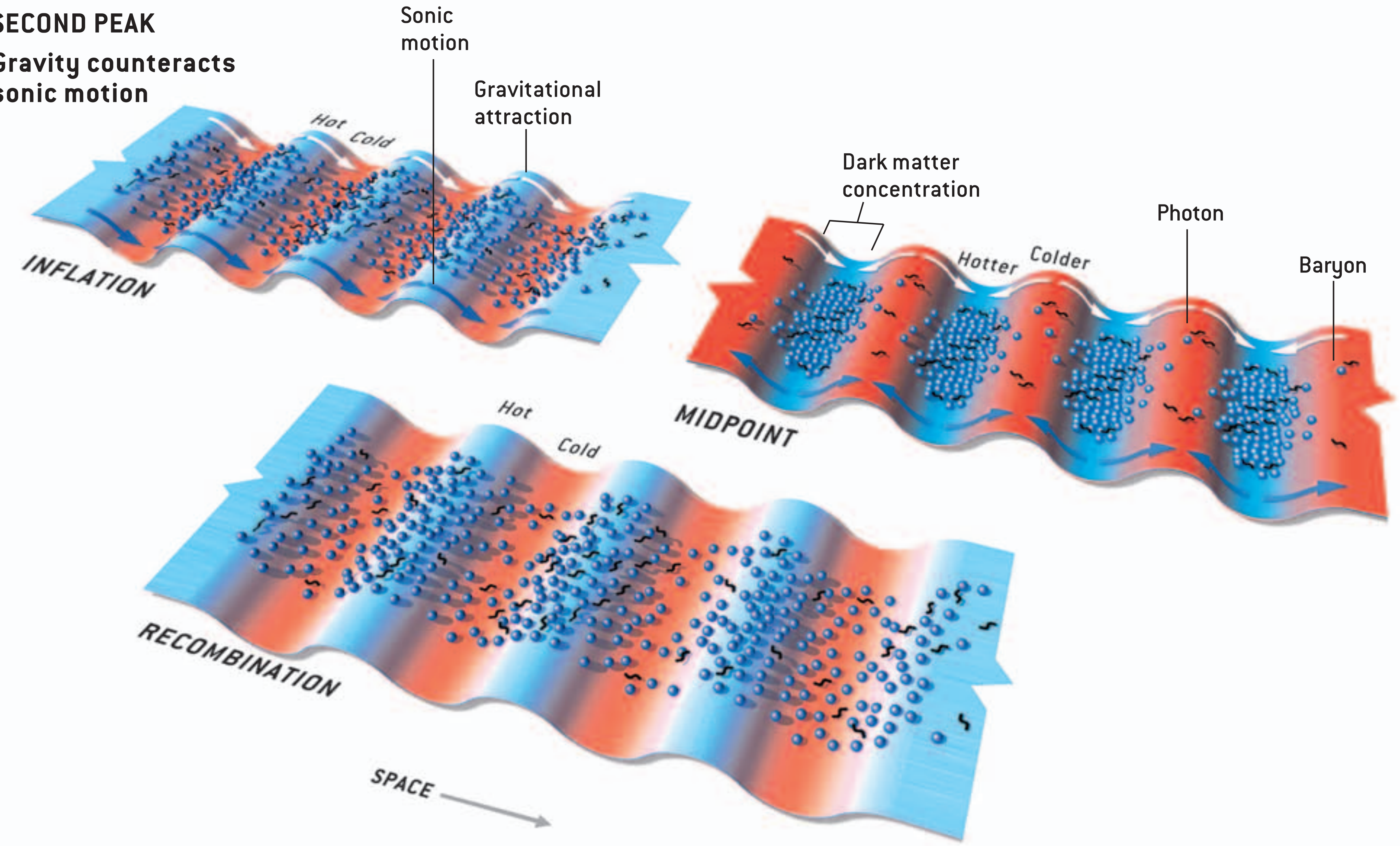
Gravity and sonic motion work together



AT SMALLER SCALES, gravity and acoustic pressure sometimes end up at odds. Dark matter clumps corresponding to a second-peak wave maximize radiation temperature in the troughs long before recombination. After this midpoint, gas pressure pushes baryons and photons out of the troughs (*blue arrows*) while gravity tries to pull them back in (*white arrows*). This tug-of-war decreases the temperature differences, which explains why the second peak in the power spectrum is lower than the first.

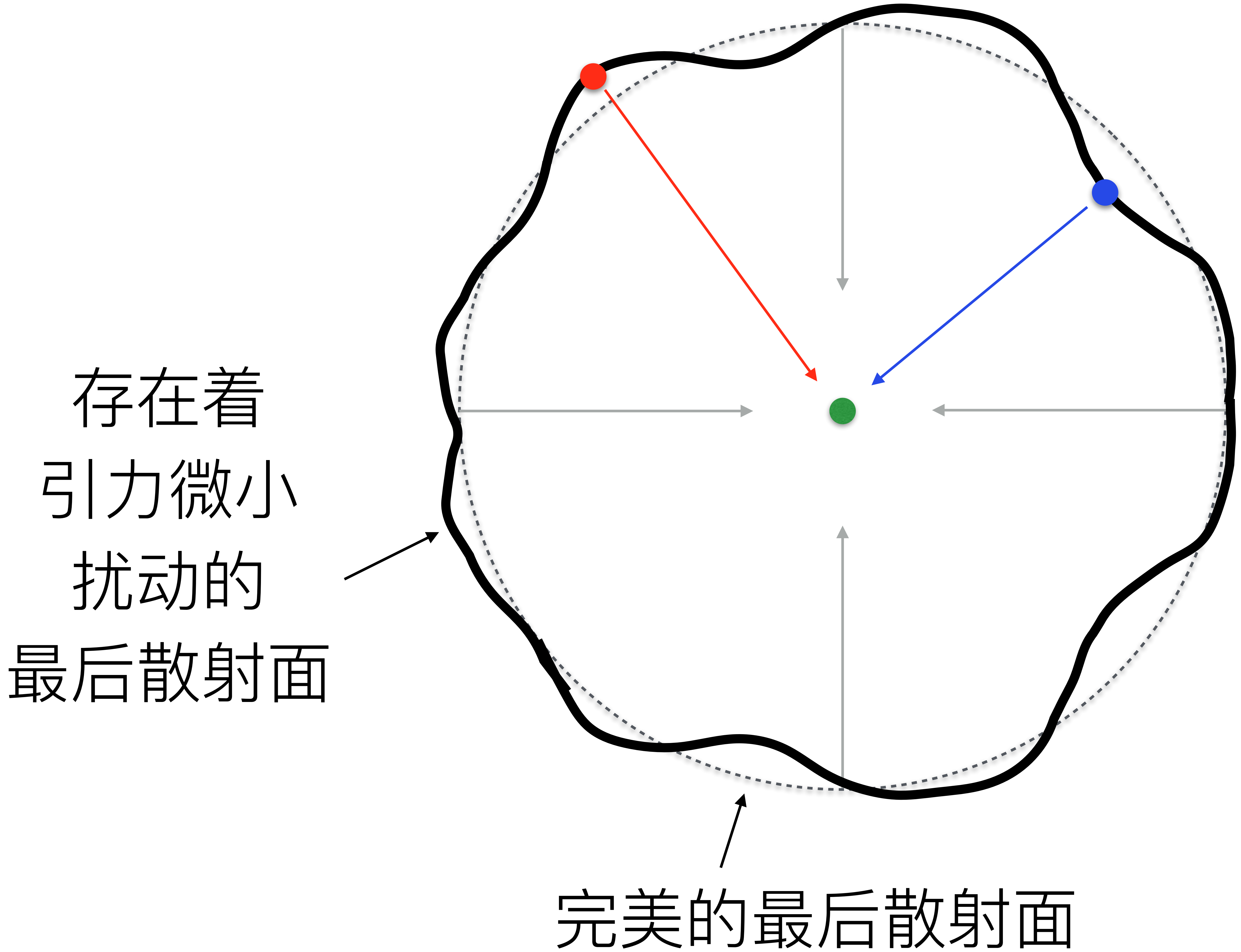
SECOND PEAK

Gravity counteracts sonic motion



CHRISTIE DESIGN

原初CMB各向异性之Sachs－Wolfe 效应



Sachs-Wolfe 68’

这就是，著名效应Sachs-Wolfe effect

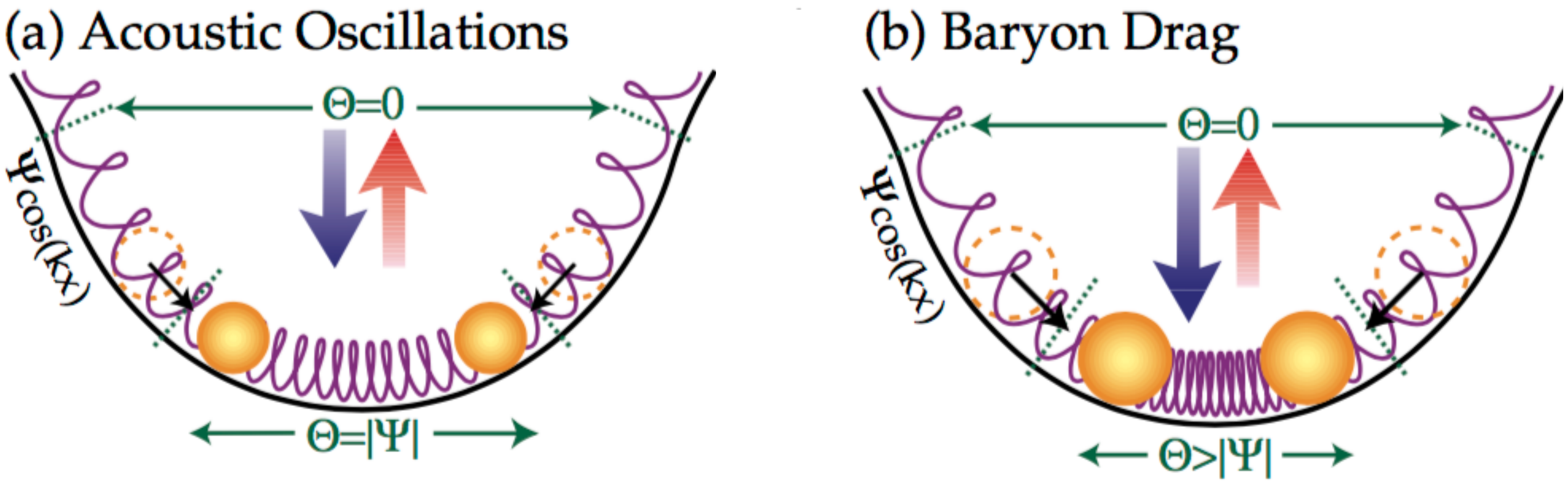
90%的CMB信号来自于此！

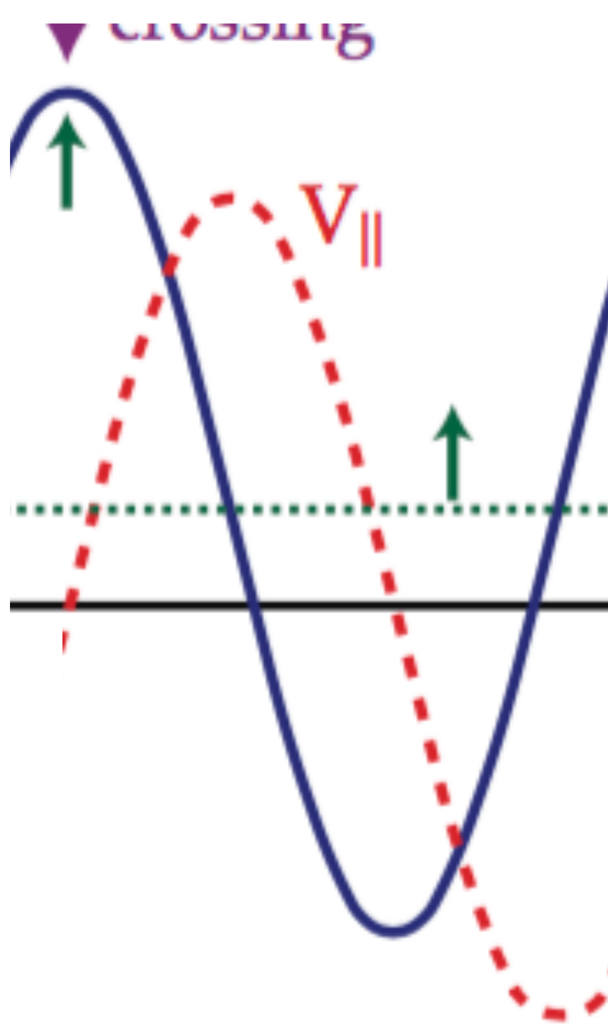
2. 重子拖曳 / baryon drag

之前的计算没有计入重子，加入重子后，由于

重子**有质量**（小球变重）、（几乎）**无压强**（弹簧弹性不变）

先不考虑，重子后势阱变深的效应（次领头阶）





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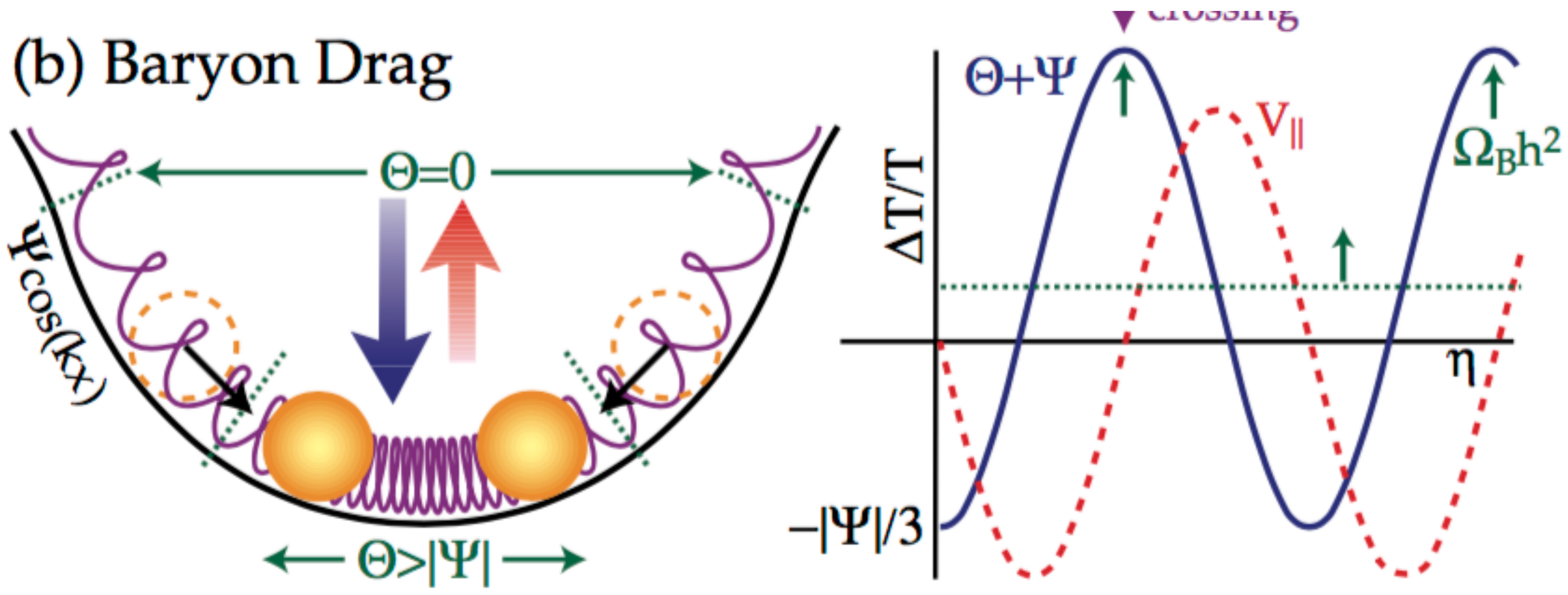


振幅变大，但为了保持IC，平衡点需要上移

IC来自暴胀理论，因此不能改变

3.多普勒效应 / Doppler effect

注意，这里讨论的是**以1/3光速**震荡的等离子体。
其相对论效应**不可忽略**，如：多普勒效应

当它沿视线方向，向我们震荡时，发生**蓝移**，
光子能量增加，温度升高；反之，**红移**

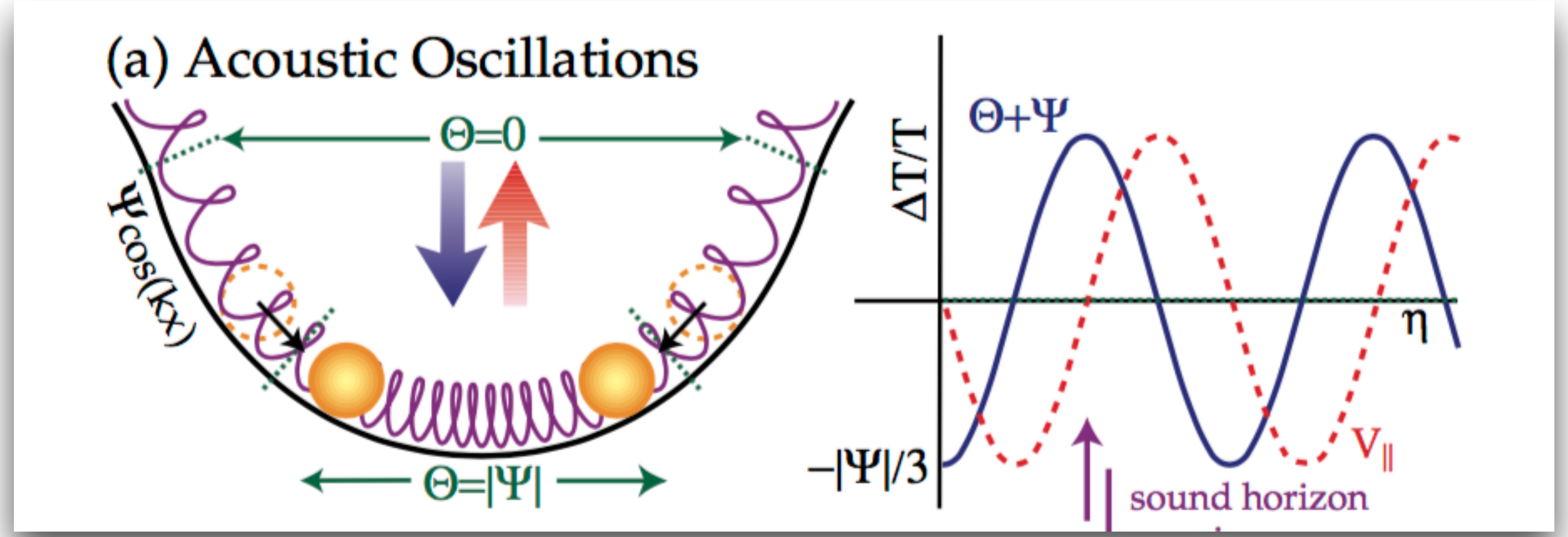
$$f^{obs} = (1 + \frac{v}{c}) f^{rest}$$

$$f \propto T$$

Wien displacement law

$$\left.\frac{\Delta T}{T}\right|_{doppler} = \frac{v}{c}$$

该效应，对光子温度的贡献为： $\Psi \sin(ks)/3$



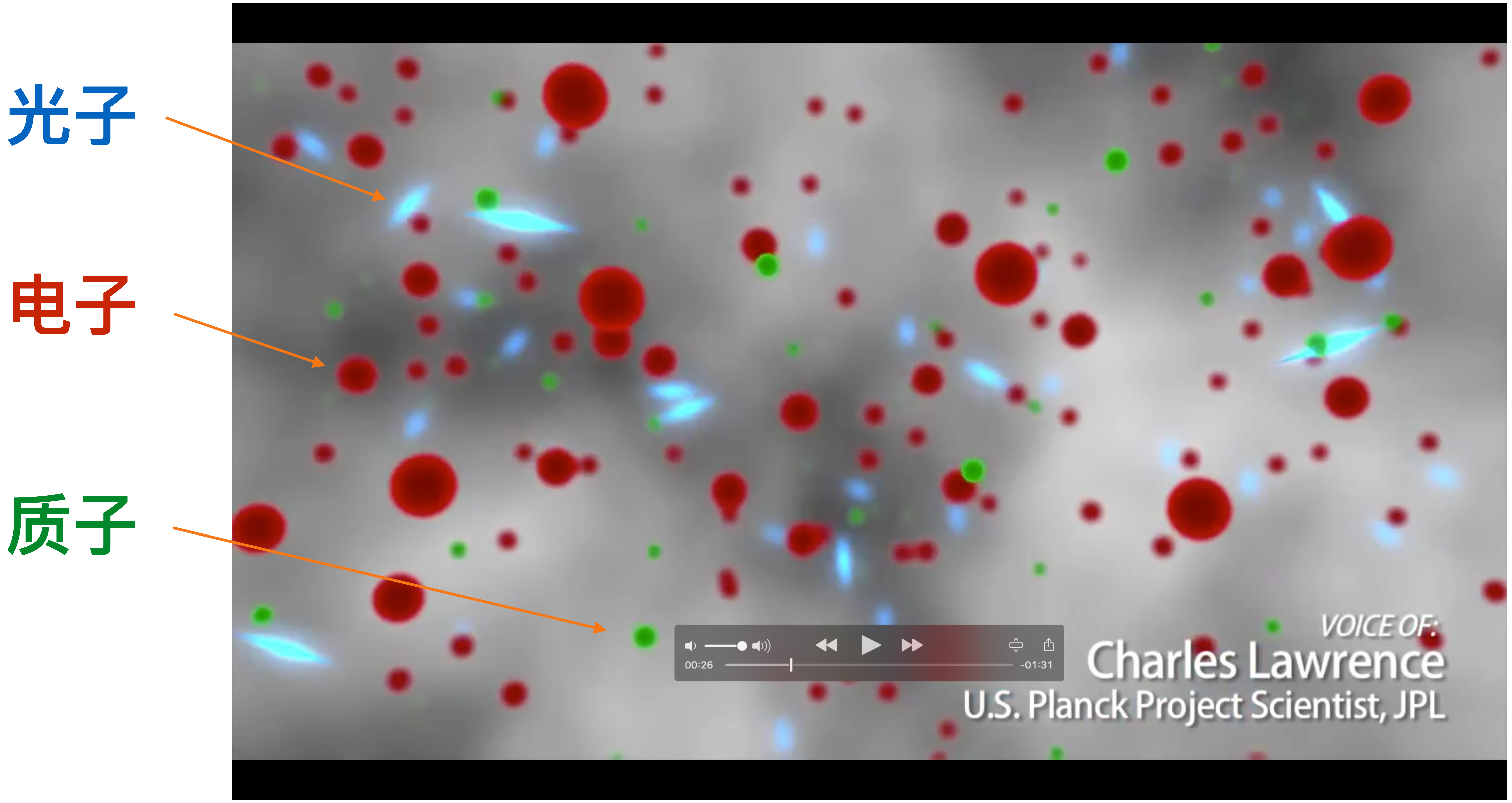
蓝线为等离子体**能量密度**扰动对光子温度的贡献

红线为等离子体**速度**扰动对光子温度的贡献

二者，相差一个 Pi / 2的相位！

4. 光子弥散 / Diffusion

光子，与电子不断碰撞。在两次碰撞之间，光子自由穿行。
这个距离被称作，光子的平均自由程 / mean free path



Tight coupling is not that perfect,

光子在电子之间的随机行走，可以使得等离子体的**冷热部分相互混合**，**从而抹平温度扰动**，这称为**光子弥散**

波长比光子平均自由程小的，光子温度扰动，
被光子弥散效应e指数压低 （小尺度效应）

在最后散射发生之后，光子平均自由程近似于无穷大，
即，宇宙38万年时刻的信息，经过138亿年的雨雪风霜，
几乎毫无损失地保留到现在！

这就是，之前我们所说的“baby face”

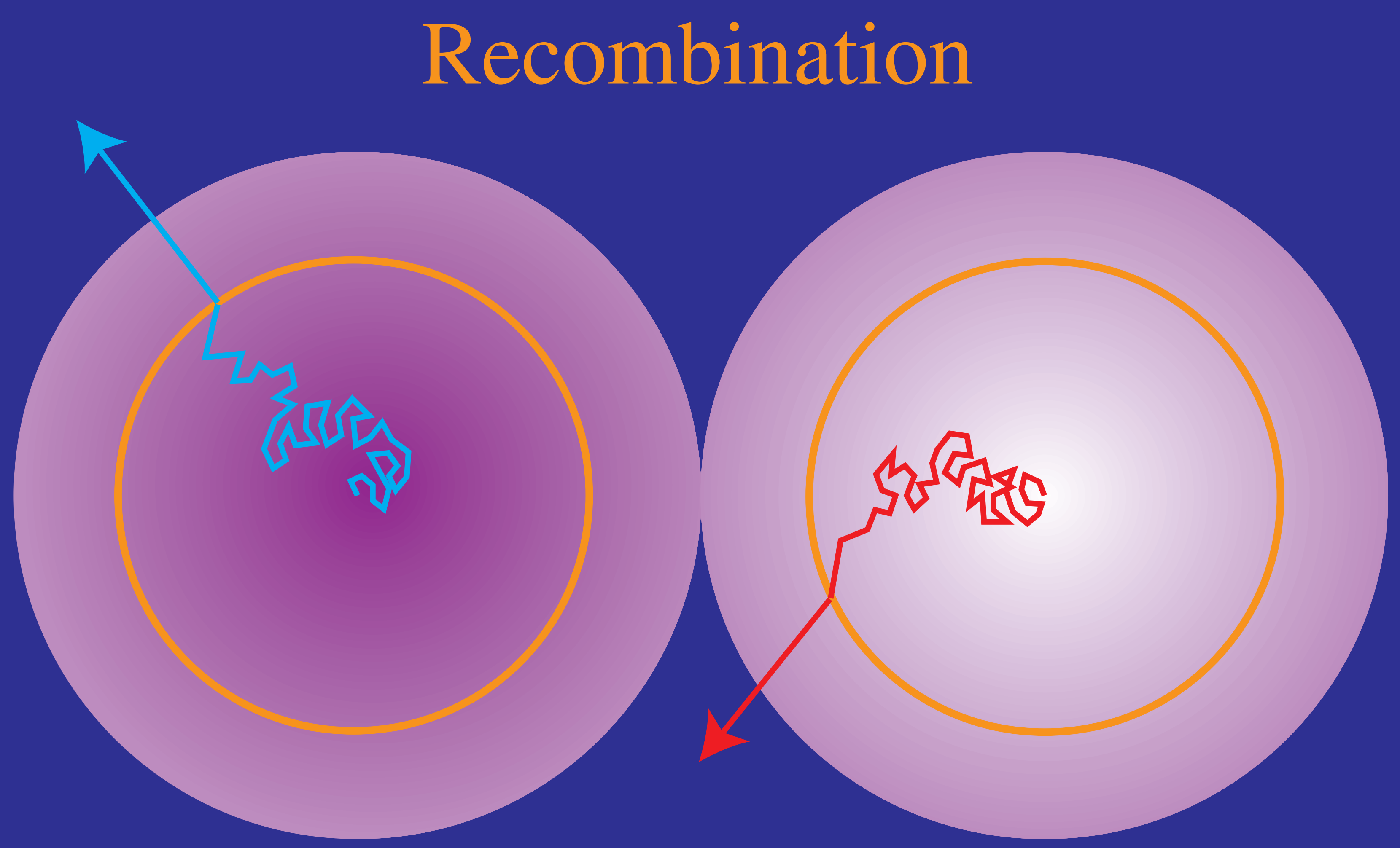
Dissipation / Diffusion Damping

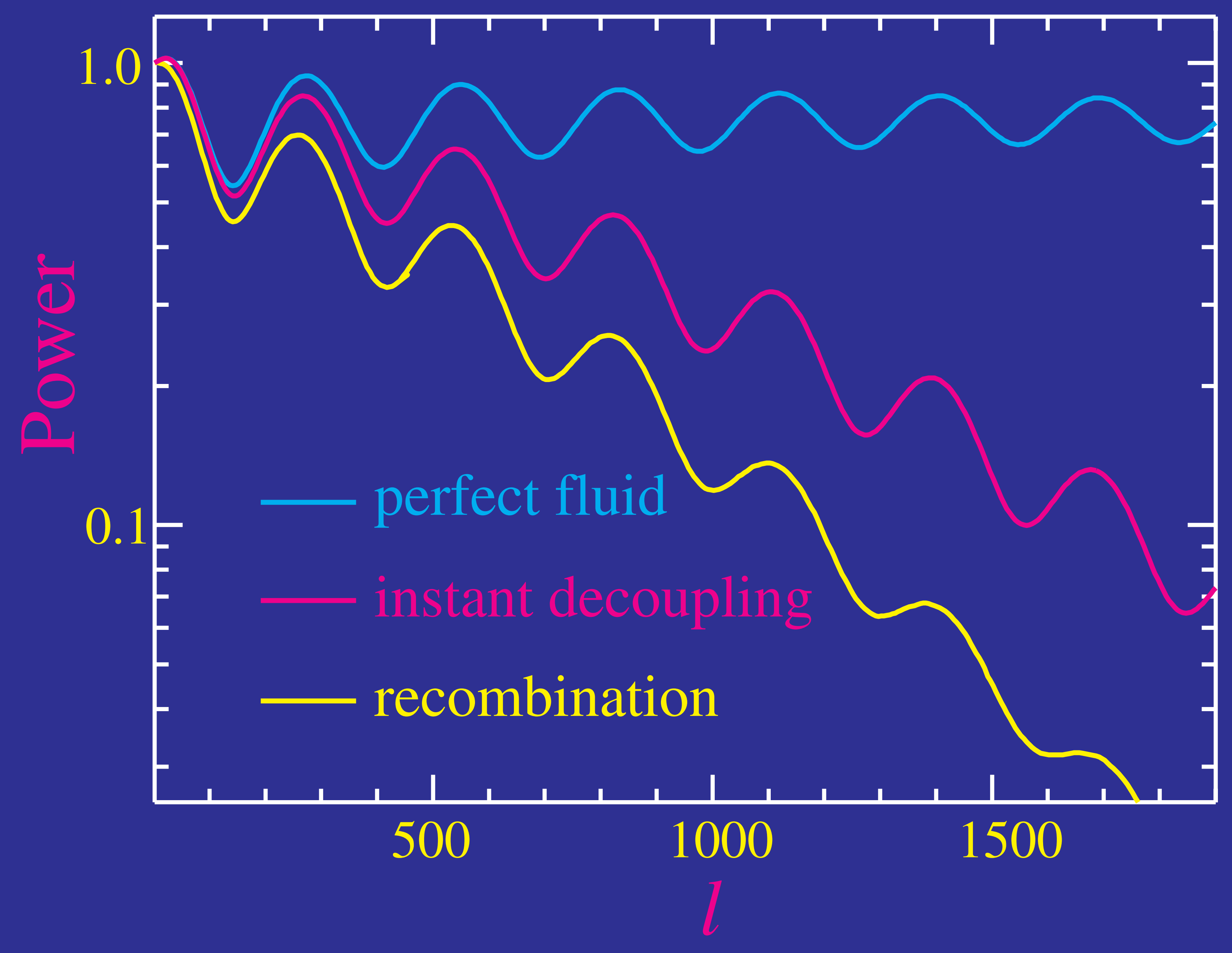
[credit: W. Hu]

Q: Why sqrt{N}?

- Imperfections in the coupled fluid → mean free path λ_C in the baryons
- Random walk over diffusion scale: $\lambda_D \sim \lambda_C\sqrt{N} \sim \sqrt{\lambda_C \eta} \gg \lambda_C$
- Rapid increase at recombination as mfp ↑

- Robust physical scale for angular diameter distance test (Ω_K, Ω_Λ)





Silk (1968); Hu & White (1996)

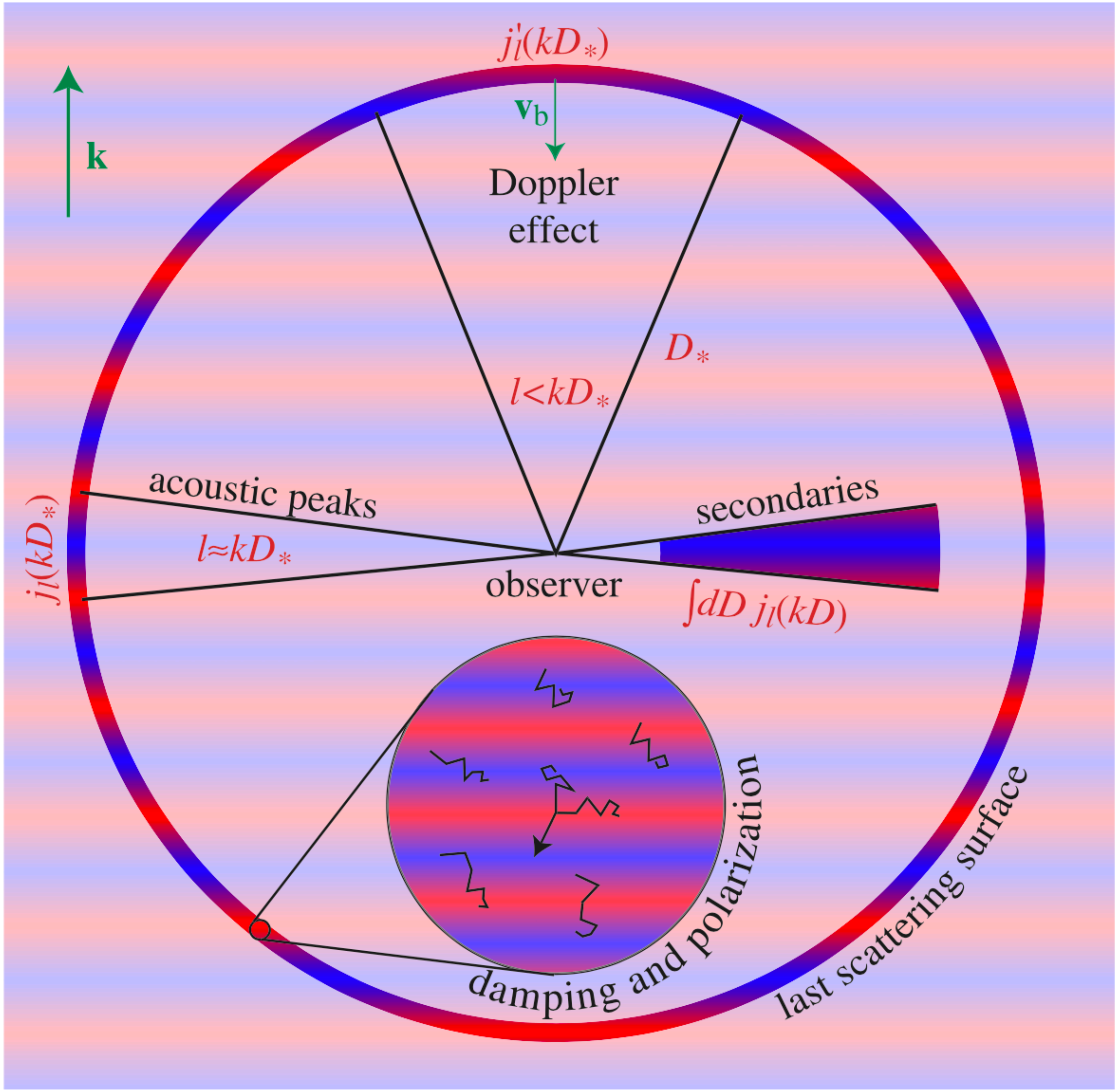
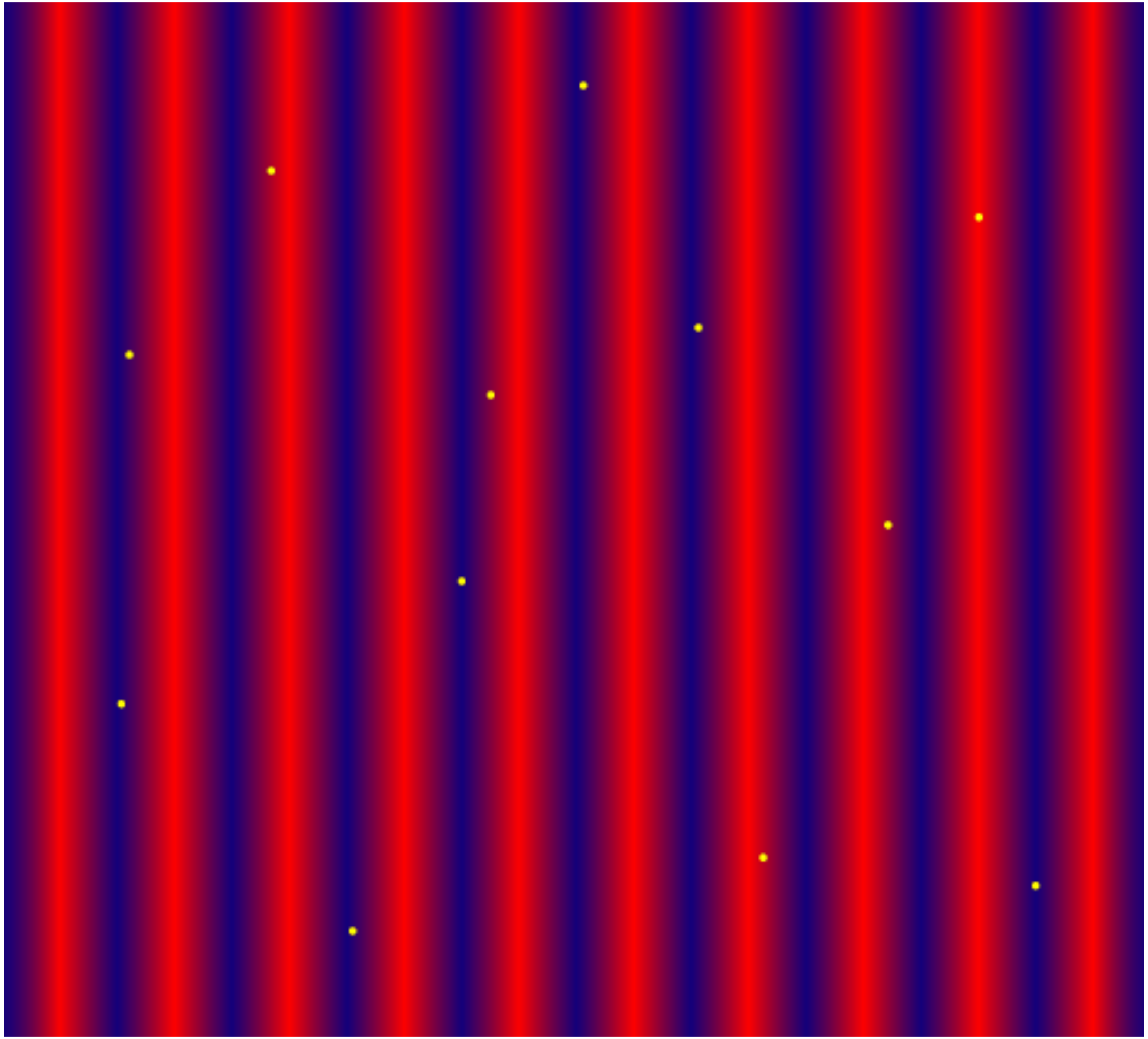


Plate 3: Integral approach. CMB anisotropies can be thought of as the line-of-sight projection of various sources of plane wave temperature and polarization fluctuations: the acoustic effective temperature and velocity or Doppler effect (see §3.8), the quadrupole sources of polarization (see §3.7) and secondary sources (see §4.2, §4.3). Secondary contributions differ in that the region over which they contribute is thick compared with the last scattering surface at recombination and the typical wavelength of a perturbation.

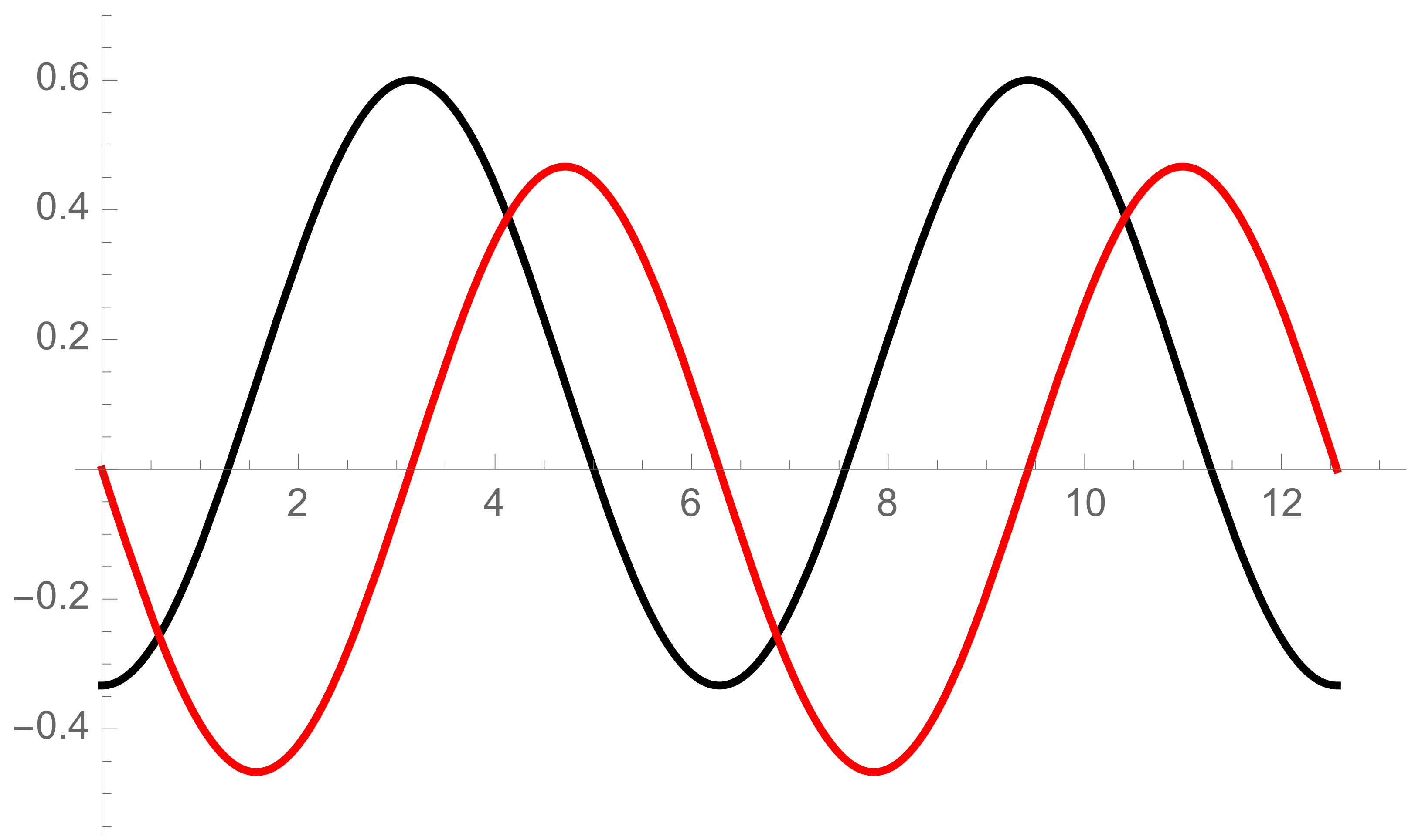
[Hu & Dodelson
Annu. Rev. Astron. and Astrophys. 2002]

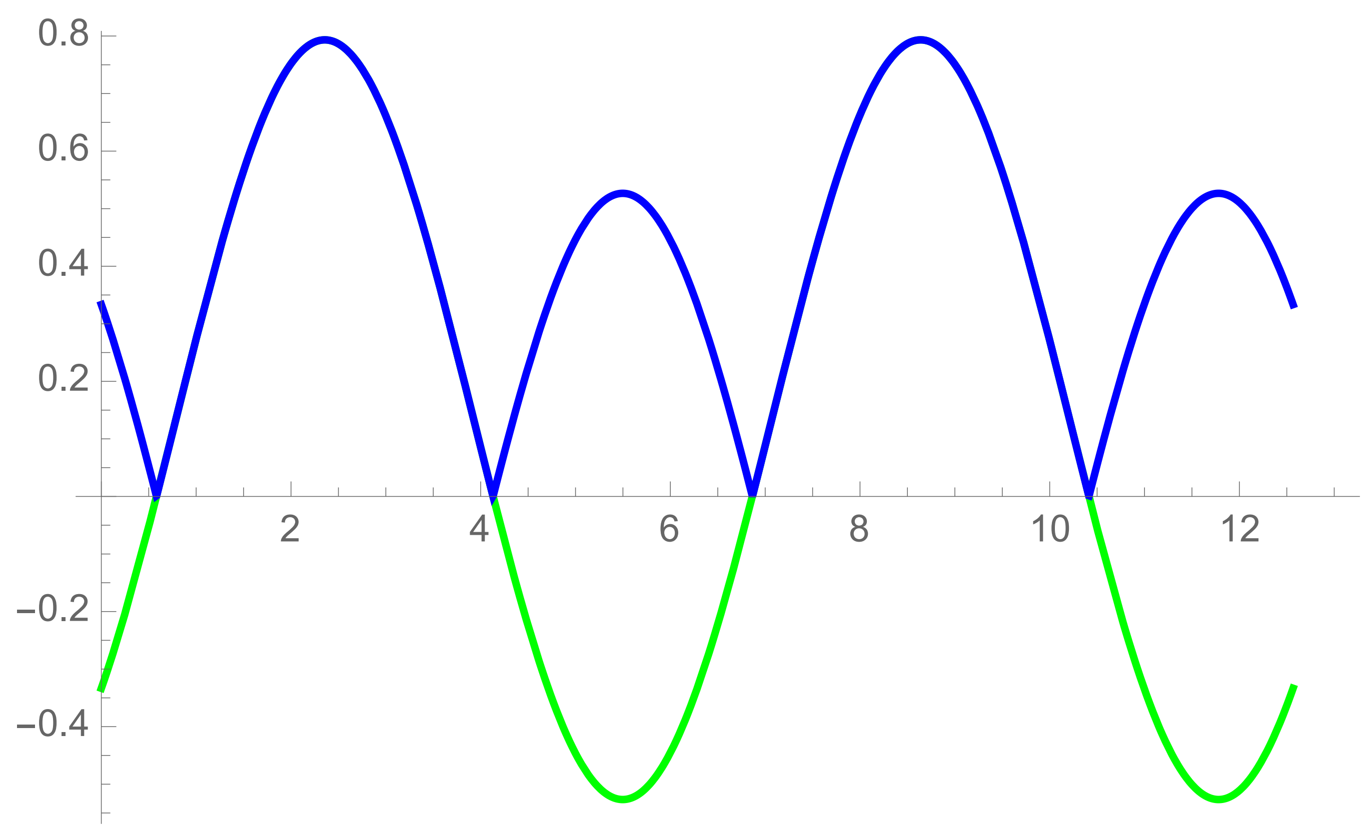
Photons transfer heat

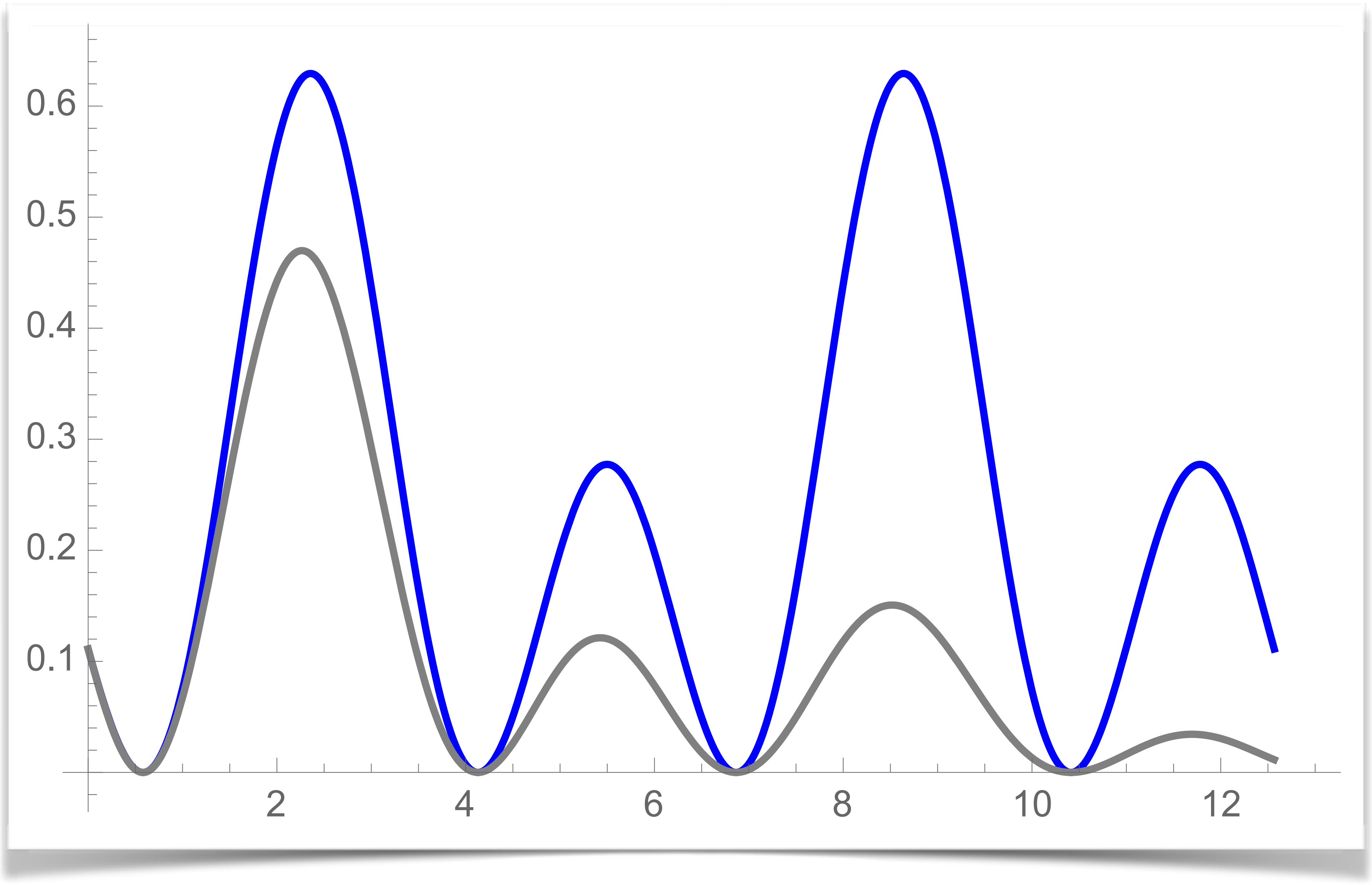
between hot and cold spots



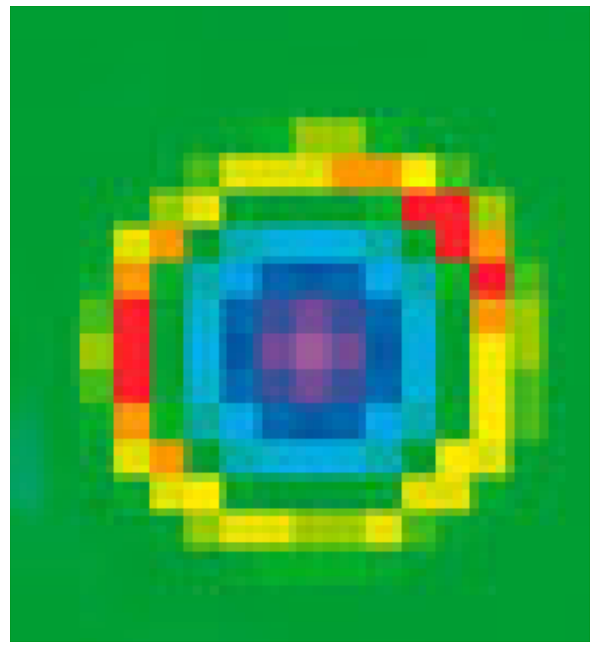
[credit: W. Hu]


0.6
0.4
0.2
−0.2
−0.4
2
4
6
8
10
12


0.8
0.6
0.4
0.2
−0.2
−0.4
2
4
6
8
10
12

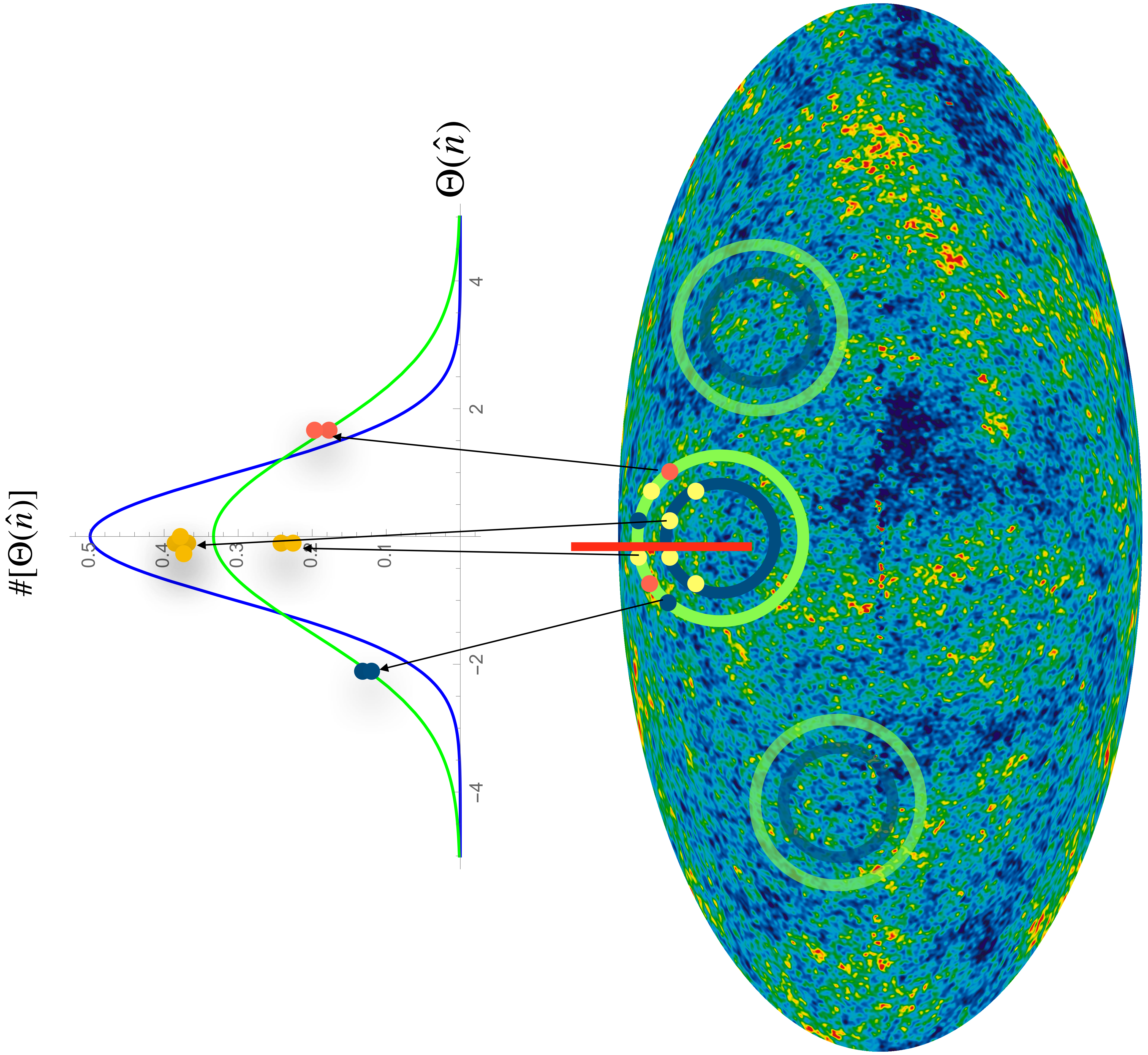

0.6
0.5
0.4
0.3
0.2
0.1
2
4
6
8
10
12

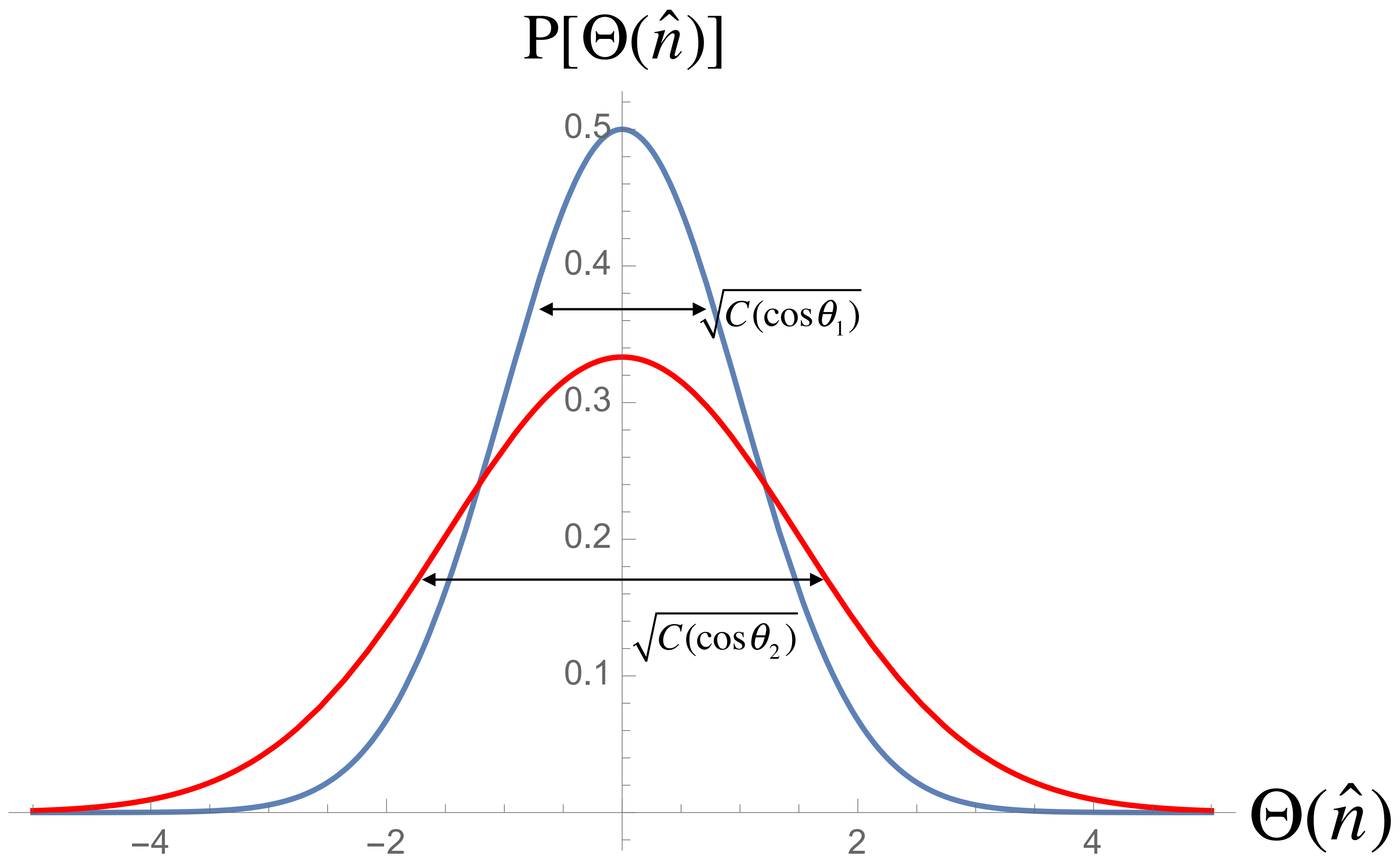
$$\Theta(\theta,\varphi)=\frac{\delta T}{T}(\theta,\varphi)$$ **is a Gaussian random field on the 2D sphere**



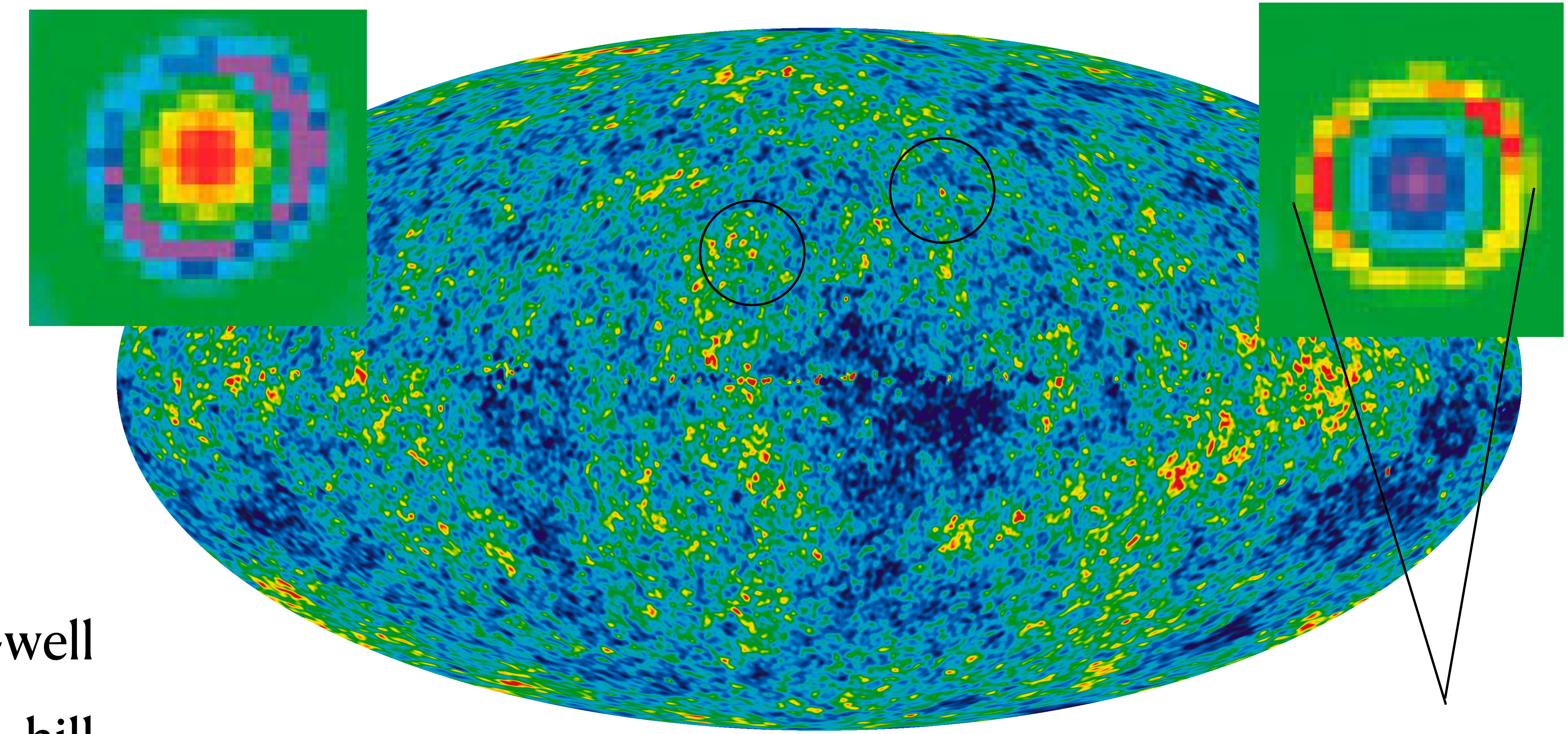
we will study the angular distribution of $\Theta(\theta,\varphi)$

$$<\Theta(\hat{n})\Theta(\hat{n}')>=\int d\hat{n}\int d\hat{n}'\Theta(\hat{n})\Theta(\hat{n}')=C(\hat{n}\bullet\hat{n}')=C(\cos\theta)$$ **statistical isotropy**





注意，对于单个扰动而言，其分布是各向异性的
但，扰动在各个方向上的平均值，则是各向同性的，
这称为**统计各向同性**



hot spot~well

cold spot~hill

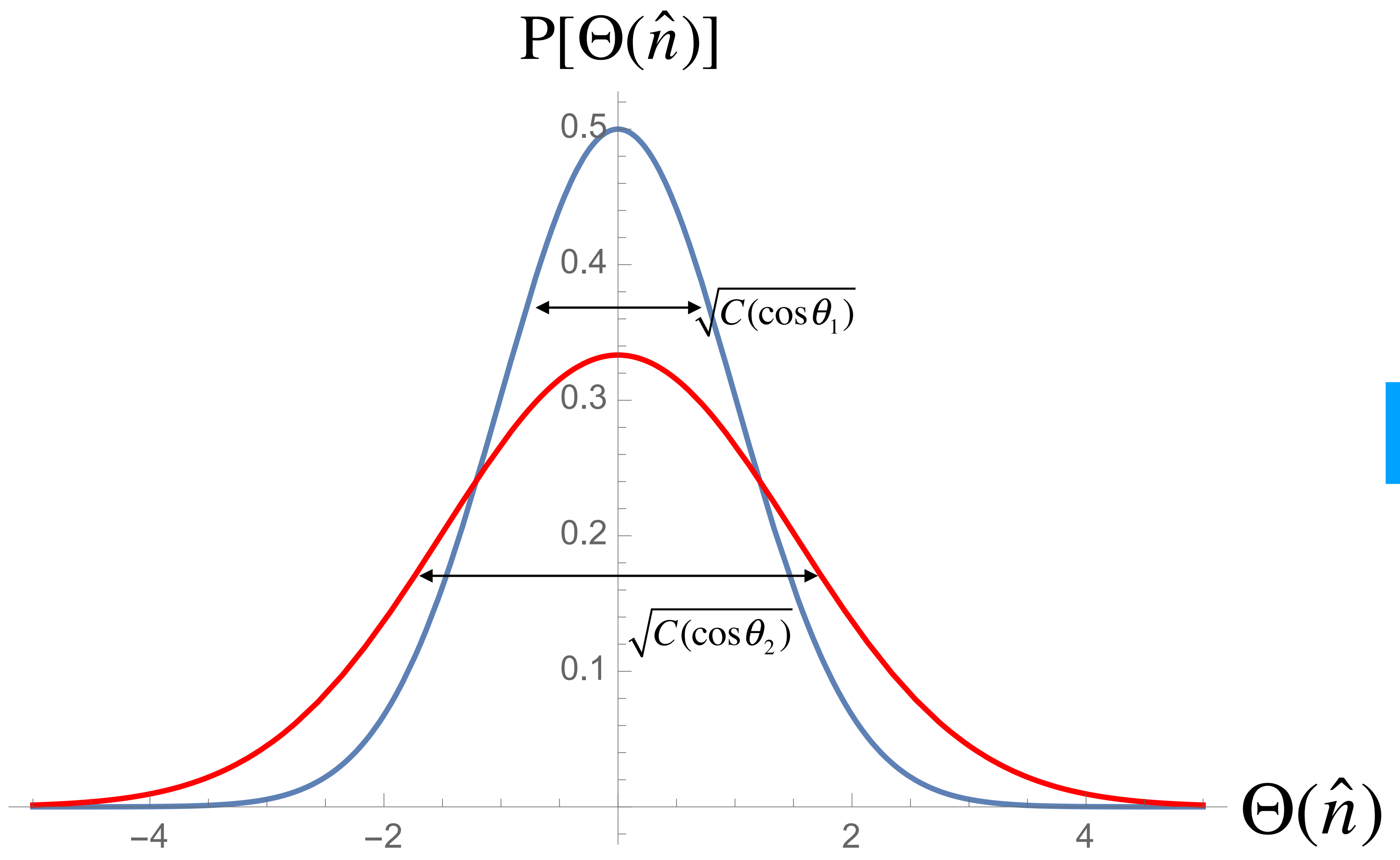
A smarter way

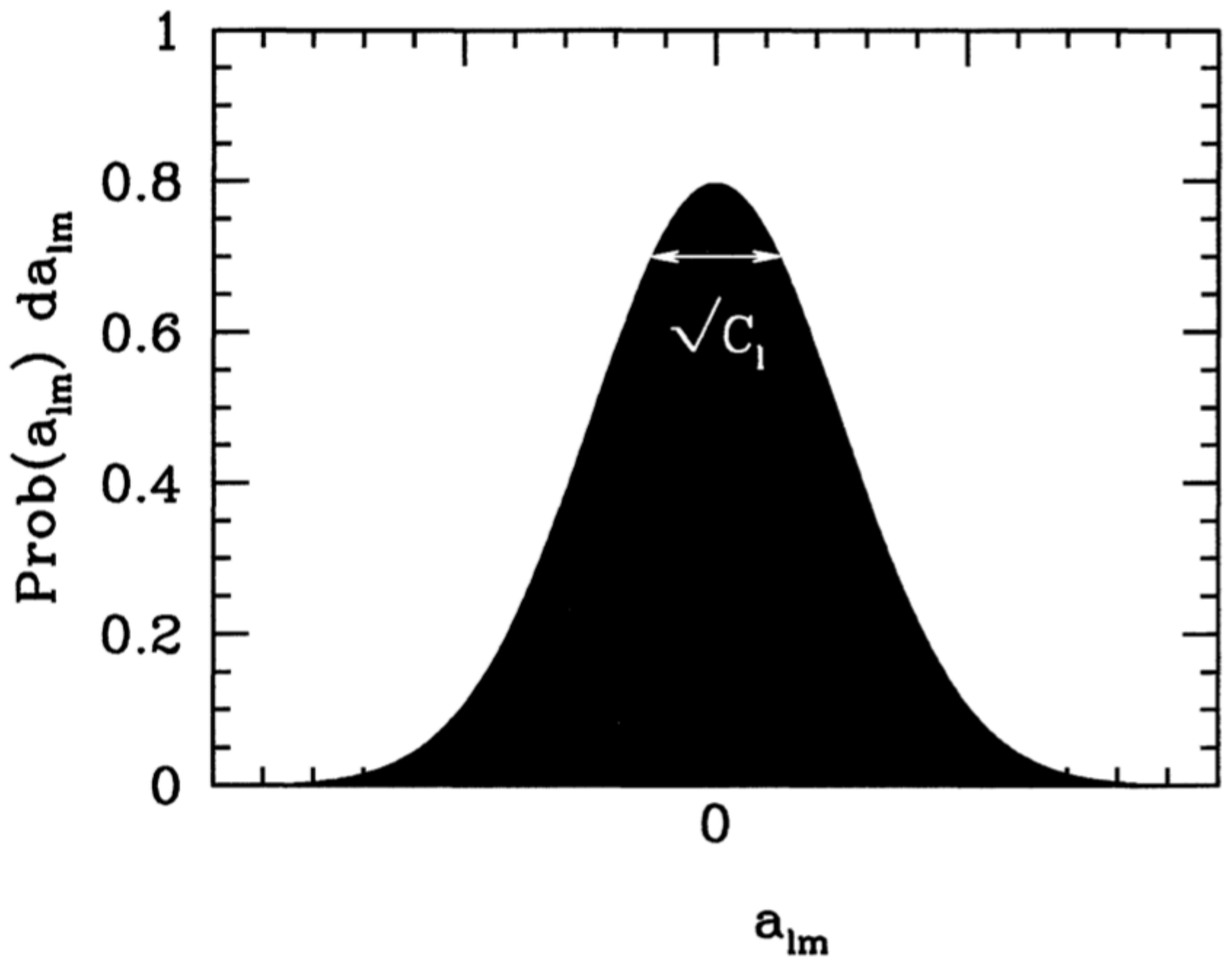
$$\Theta(\hat{n}) = \sum_{\ell,m} a_{\ell m} Y_{\ell m}(\hat{n})$$

$$\Theta(\hat{n}) \Rightarrow a_{\ell m}$$

gaussian random field

$$\left\langle \frac{\Delta T}{T}(\mathbf{n}) \frac{\Delta T}{T}(\mathbf{n}') \right\rangle_{\mathbf{n}\cdot\mathbf{n}'=\mu} = \sum_{\ell,\ell',m,m'} \langle a_{\ell m} \cdot a^*_{\ell' m'} \rangle\, Y_{\ell m}(\mathbf{n}) Y^*_{\ell m}(\mathbf{n}') = \sum_{\ell} C_\ell \underbrace{\sum_{m=-\ell}^{\ell} Y_{\ell m}(\mathbf{n}) Y^*_{\ell m}(\mathbf{n}')}_{\frac{2\ell+1}{4\pi} P_\ell(\mathbf{n}\cdot\mathbf{n}')} = \frac{1}{4\pi} \sum_{\ell} (2\ell+1) C_\ell P_\ell(\mu),$$





our major task is to calculate/measure C_ℓ

derivation of cosmic variance

$$b_{\ell m} = a_{\ell m} + n_{\ell m}$$

$$\hat{C}_\ell = \frac{1}{2\ell+1} \sum_m b^*_{\ell m} b_{\ell m}$$

$$= \frac{1}{2\ell+1} \sum_m (a^*_{\ell m} + n^*_{\ell m})(a_{\ell m} + n_{\ell m})$$

$$= \frac{1}{2\ell+1} \sum_m (a^*_{\ell m} a_{\ell m} + n^*_{\ell m} n_{\ell m})$$

$$\langle \hat{C}_\ell \rangle = \frac{1}{2\ell+1} \sum_m \Big(\underbrace{\langle a^*_{\ell m} a_{\ell m} \rangle}_{C_\ell} + \underbrace{\langle n^*_{\ell m} n_{\ell m} \rangle}_{N_\ell} \Big)$$

$$= \frac{1}{2\ell+1} \sum_m (C_\ell + N_\ell)$$

$= \underline{C_\ell + N_\ell}$ ← expectation.

Variance

$$\langle (\hat{C}_\ell - \langle \hat{C}_\ell \rangle)(\hat{C}_\ell - \langle \hat{C}_\ell \rangle) \rangle$$

$$= \langle \hat{C}_\ell \hat{C}_\ell \rangle - 2 \langle \hat{C}_\ell \rangle \langle \hat{C}_\ell \rangle + (\langle \hat{C}_\ell \rangle)^2$$

$$= \langle \hat{C}_\ell \hat{C}_\ell \rangle - (\langle \hat{C}_\ell \rangle)^2$$

$$(\langle \hat{C} \rangle)^2 = (C_\ell + N_\ell)^2 = C_\ell^2 + N_\ell^2 + 2 C_\ell N_\ell \equiv \square$$

$$= \frac{1}{(2\ell+1)^2} \sum_{m,n} \langle b^*_{\ell m} b_{\ell m} b^*_{\ell n} b_{\ell n} \rangle - \square$$

$$= \frac{1}{(2\ell+1)^2} \sum_{m,n} \Big[\langle b^*_{\ell m} b_{\ell m} \rangle \langle b^*_{\ell n} b_{\ell n} \rangle + \langle b^*_{\ell m} b^*_{\ell n} \rangle \langle b_{\ell m} b_{\ell n} \rangle + \langle b^*_{\ell m} b_{\ell n} \rangle \langle b^*_{\ell n} b_{\ell m} \rangle \Big] - \square$$

$$= \frac{1}{(2\ell+1)^2} \sum_{m,n} \Big[(C_\ell + N_\ell)^2 + \langle b^*_{\ell m} b_{\ell -n} \rangle \langle b^*_{\ell m} b_{\ell n} \rangle + \langle b^*_{\ell m} b_{\ell n} \rangle \langle b^*_{\ell n} b_{\ell m} \rangle \Big] - \square$$

$$= \frac{1}{(2\ell+1)^2} \sum_{m,n} \Big[\cancel{(2\ell+1)^2 (C_\ell + N_\ell)^2} + \langle b^*_{\ell m} b_{\ell -n} \rangle \langle b^*_{\ell -m} b_{\ell n} \rangle + \langle b^*_{\ell m} b_{\ell n} \rangle \langle b^*_{\ell n} b_{\ell m} \rangle \Big] \cancel{- \square}$$

$$= \sum_{m,n} \Big[\langle b^*_{\ell m} b_{\ell -n} \rangle \langle b^*_{\ell -m} b_{\ell n} \rangle + \langle b^*_{\ell m} b_{\ell n} \rangle \langle b^*_{\ell n} b_{\ell m} \rangle \Big] \frac{1}{(2\ell+1)^2}$$

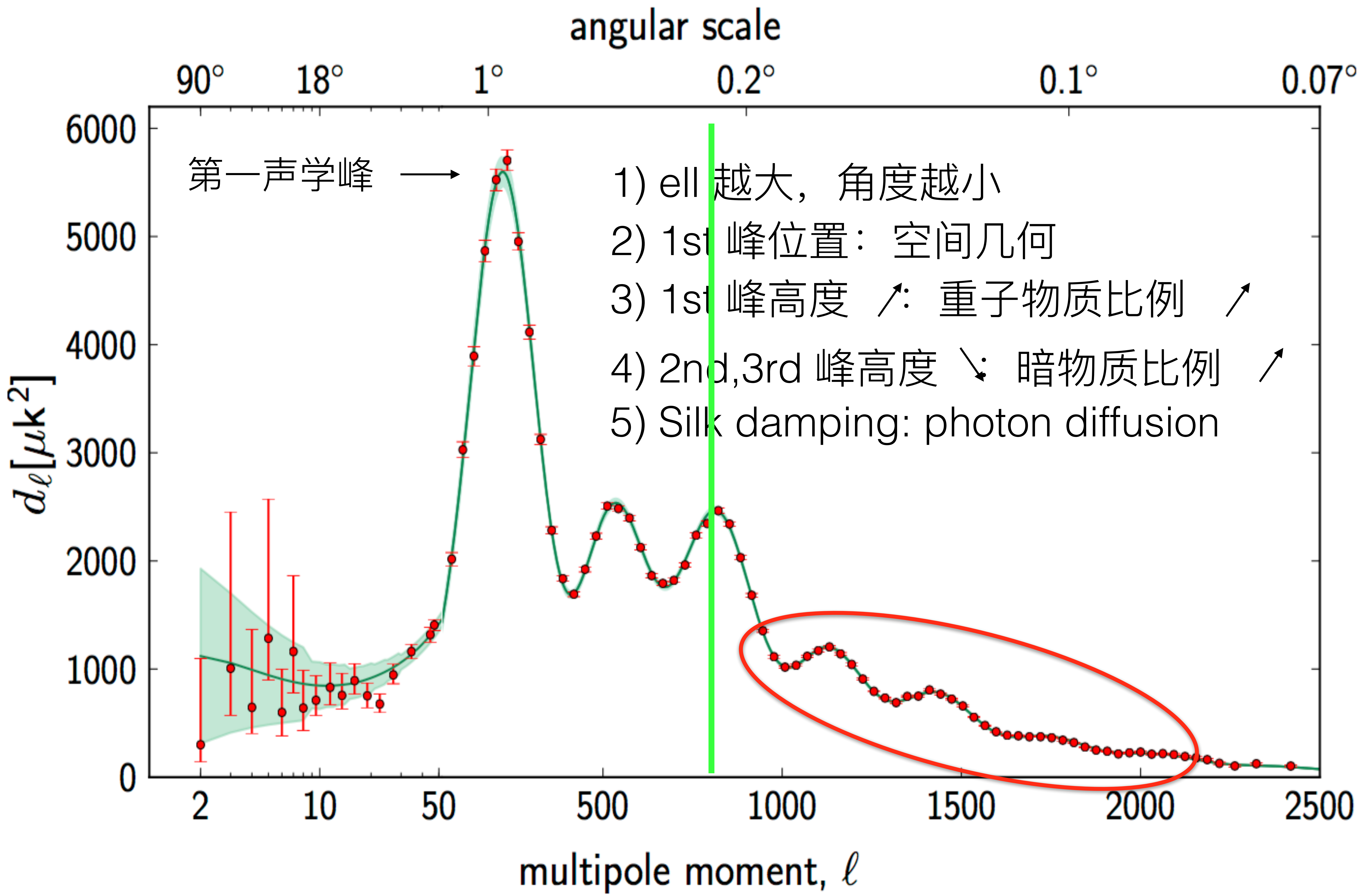
$$= \frac{1}{(2\ell+1)^2} \sum_{m,n} \Big[\delta_{m,-n} (C_\ell + N_\ell)^2 + \delta_{m,n} (C_\ell + N_\ell)^2 \Big]$$

$$= \frac{1}{(2\ell+1)^2} \Big[\sum_m (C_\ell + N_\ell)^2 + \sum_m (C_\ell + N_\ell)^2 \Big]$$

$$= \frac{1}{(2\ell+1)^2} \Big[2 \cdot (2\ell+1) (C_\ell + N_\ell)^2 \Big]$$

$= \frac{2}{2\ell+1} (C_\ell + N_\ell)^2$ ← covariance $\square$

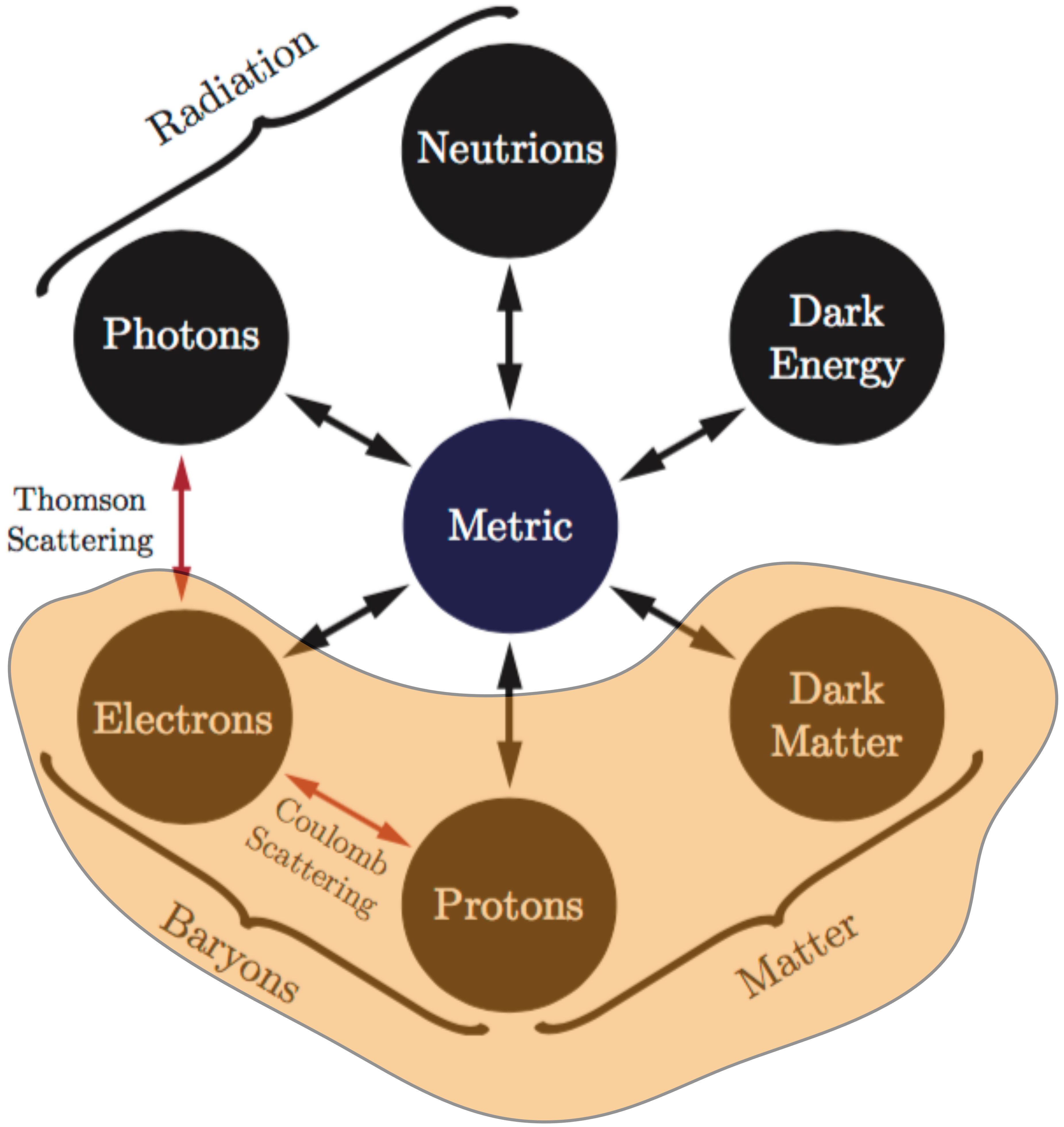
$\frac{2}{2\ell+1} C_\ell^2$ is the cosmic variance $\square$



angular scale
90°
18°
1°
0.2°
0.1°
0.07°
6000
5000
4000
3000
2000
1000
0
$d_\ell[\mu k^2]$
第一声学峰
1) ell 越大，角度越小
2) 1st 峰位置：空间几何
3) 1st 峰高度 ↗：重子物质比例 ↗
4) 2nd,3rd 峰高度 ↘：暗物质比例 ↗
5) Silk damping: photon diffusion
2
10
50
500
1000
1500
2000
2500
multipole moment, ℓ

非相对论性物质：
采用流体力学
的语言来描述

(该物质组分的**平均自由程**比我们**关心的尺度远小**。在我们所研究的尺度上，达到了热 / 动力学平衡。所以，不关心其粒子属性，只研究其整体行为。)



即，用为数不多的几个动力学量，如能量密度，速度，
与能量耗散相关的体 / 剪切粘滞系数，等

$$T_{\mu\nu} = \left\{ \begin{array}{cccc} -\rho & v & v & v \\ v & P+\sigma & \sigma & \sigma \\ v & \sigma & P+\sigma & \sigma \\ v & \sigma & \sigma & P+\sigma \end{array} \right\}$$

(不考虑热流交换)

$$\sigma_{ij} = -P\delta_{ij} + \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k} \right] + \eta\,\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

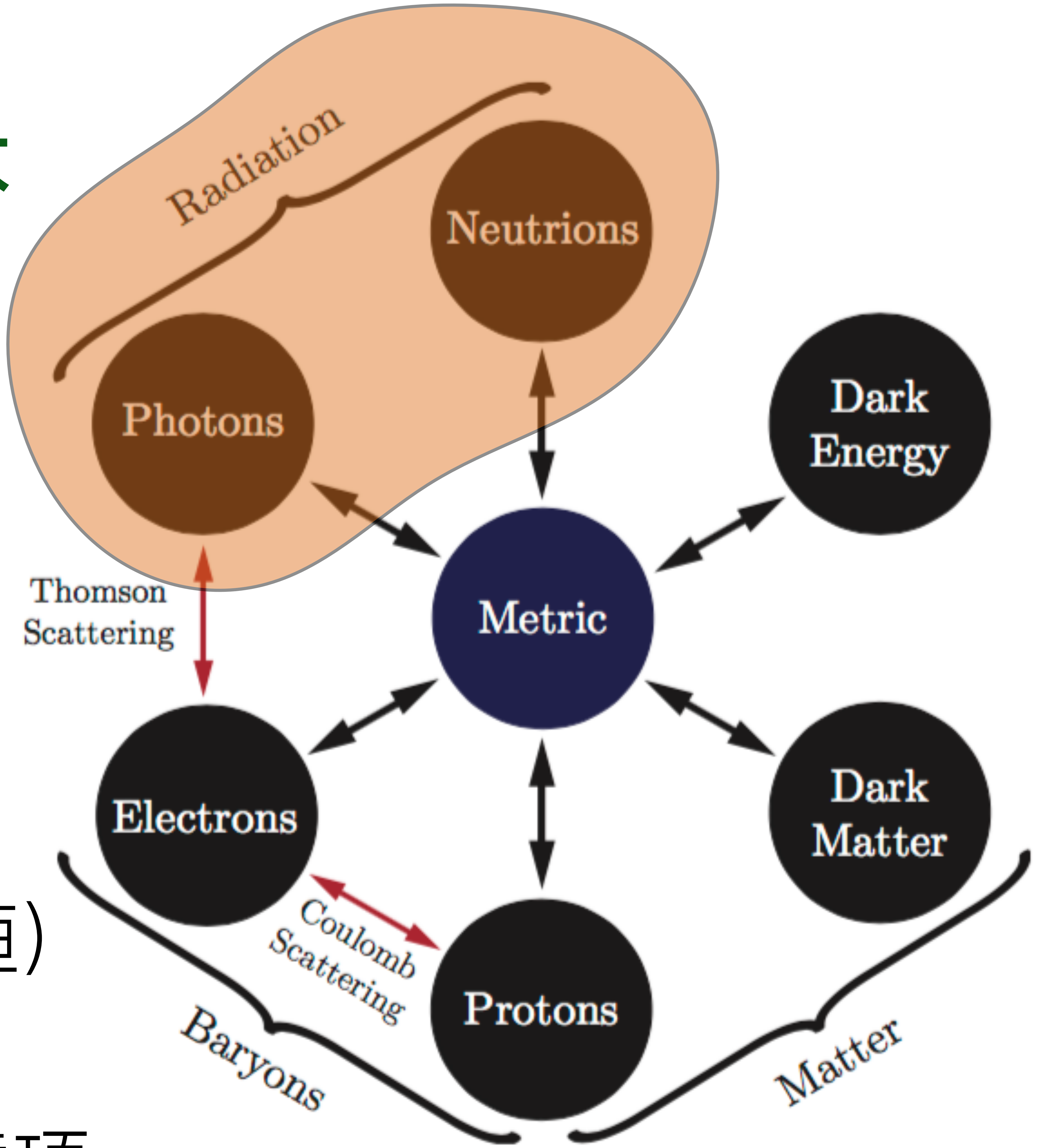
能量－动量守恒方程： $\nabla_\mu T^{\mu\nu} = 0$

$$\partial_t \rho = -\boldsymbol{\nabla}_{\boldsymbol{r}} \cdot (\rho \boldsymbol{u}) \qquad (\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}})\,\boldsymbol{u} = -\frac{\boldsymbol{\nabla}_{\boldsymbol{r}} P}{\rho} - \boldsymbol{\nabla}_{\boldsymbol{r}} \Phi$$

连续性方程 / continuity eq. 欧拉方程 / Euler eq.

相对论性物质：其**平均自由程**与
我们**关心的尺度**相比**差不多 / 更大**
他们远**未达到热 / 动力学平衡态**

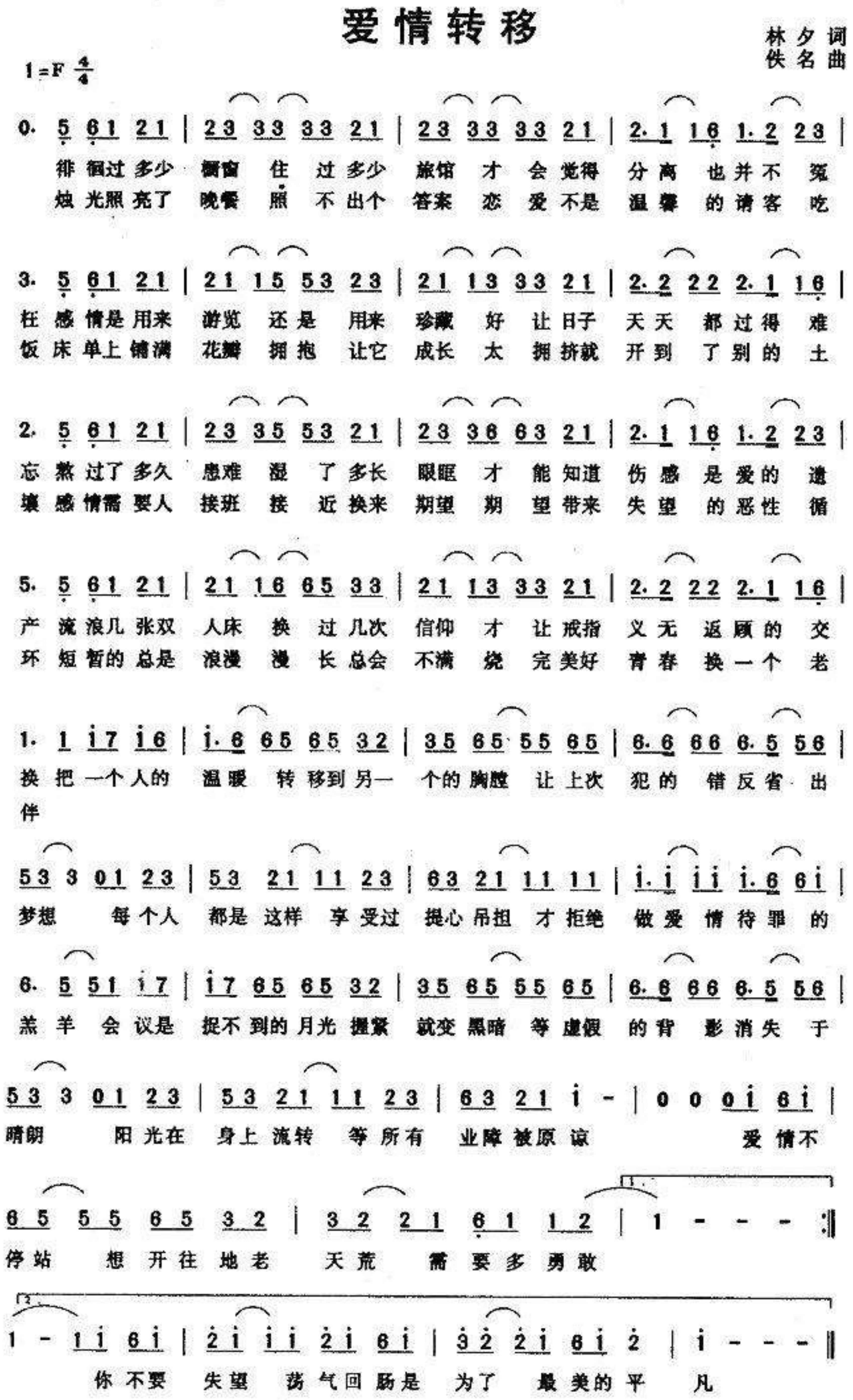
这体现了，其粒子属性，我们无法用
少数的几个热力学 / 动力学量来描述
需要借助，统计物理的方式来刻画，
即，相空间的配分函数 / partition func



刘维尔定律：（相空间中几率守恒）

$$\frac{df(\vec{x},\vec{v},t)}{dt}=C$$

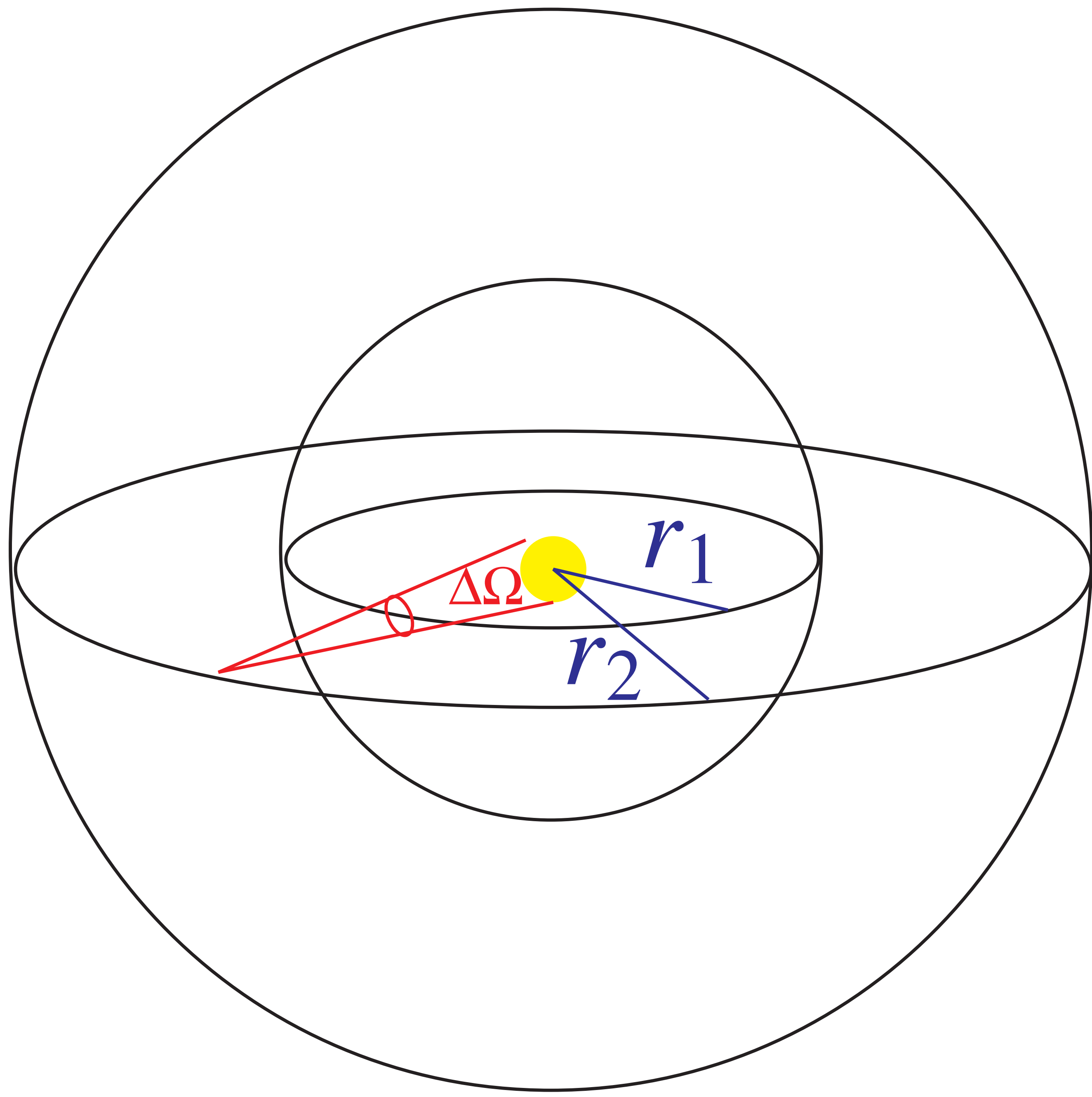
C：碰撞项
（Thomson散射）



[credit: W. Hu]

Radiative Transfer

- Radiative Transfer = change in I_ν as radiation propagates
- Simple example: how does the specific intensity of sunlight change as it propagates to the earth
- Energy conservation says



$$F(r_1)4\pi r_1^2 = F(r_2)4\pi r_2^2$$

$$F \propto r^{-2}$$

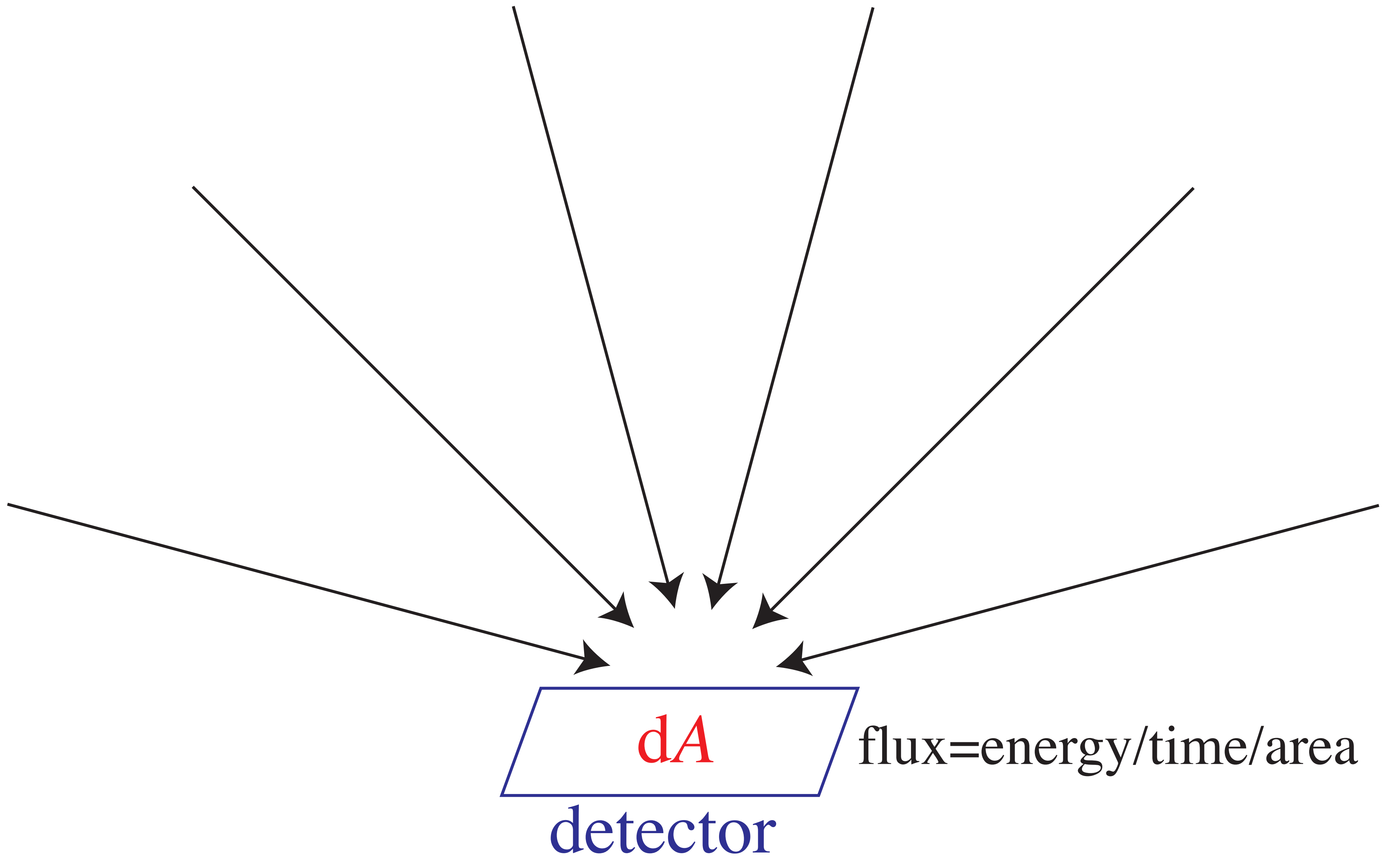
[credit: W. Hu]

Observables: Flux

- Energy Flux

$$F = \frac{dE}{dtdA}$$

- Units: erg s^{-1} cm^{-2}
- Radiation can hit detector from all angles



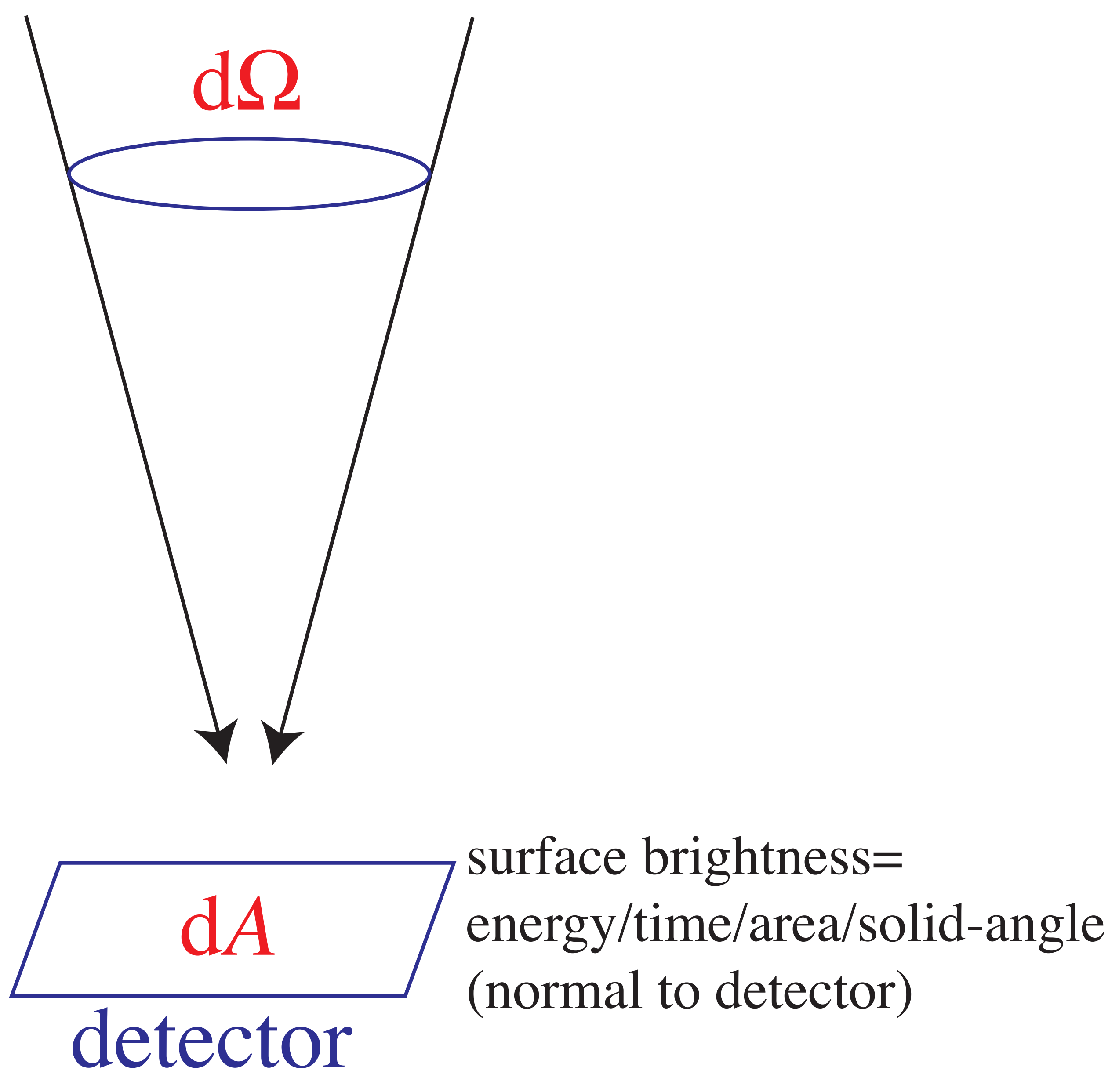
Observables: Surface Brightness

[credit: W. Hu]

- Direction: columate (e.g. pinhole) in an acceptance angle $d\Omega$ normal to $dA \to$ surface brightness

$$S(\Omega) = \frac{dE}{dt dA d\Omega}$$

- Units: erg s^{-1} cm^{-2} sr^{-1}



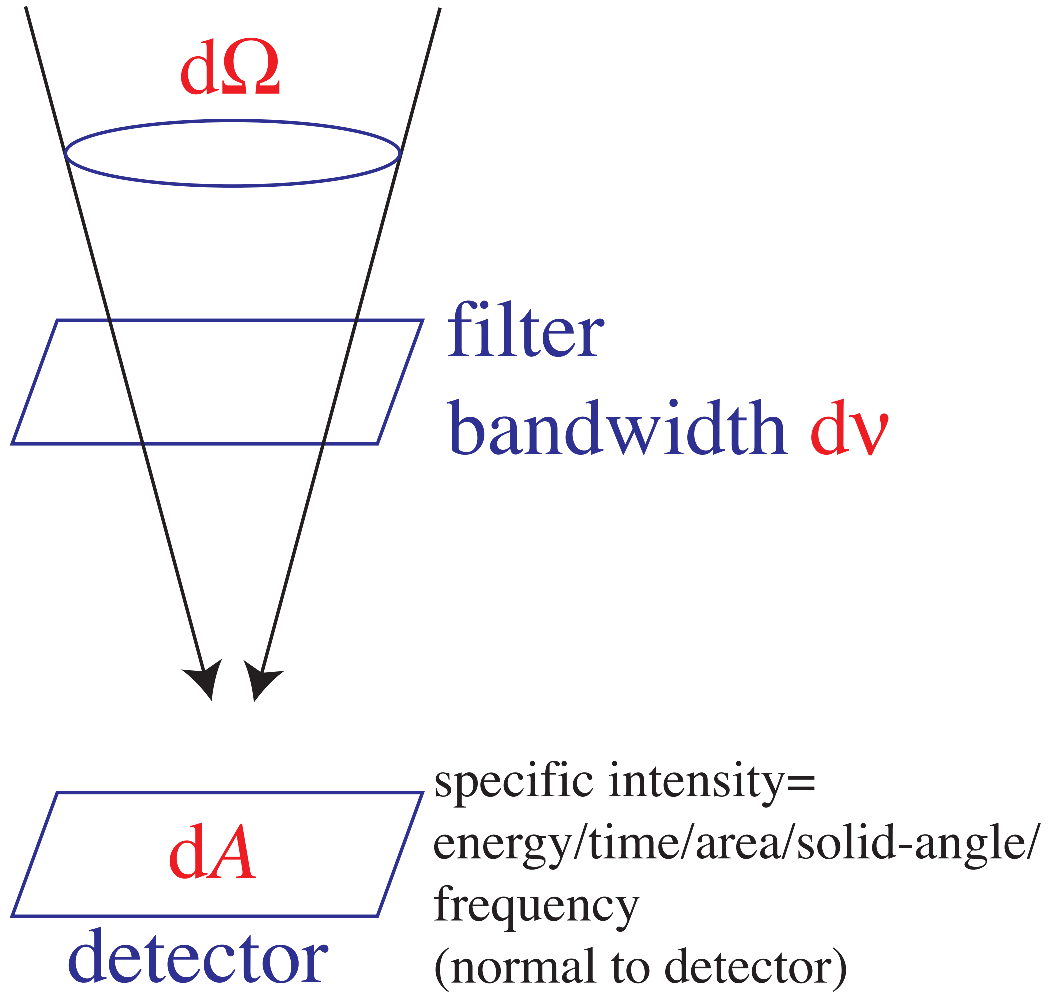
[credit: W. Hu]

Observables: Specific intensity

- Frequency: filter
  in a band of frequency
  $d\nu \to$ specific intensity

$$I_\nu = \frac{dE}{dt dA d\Omega d\nu}$$

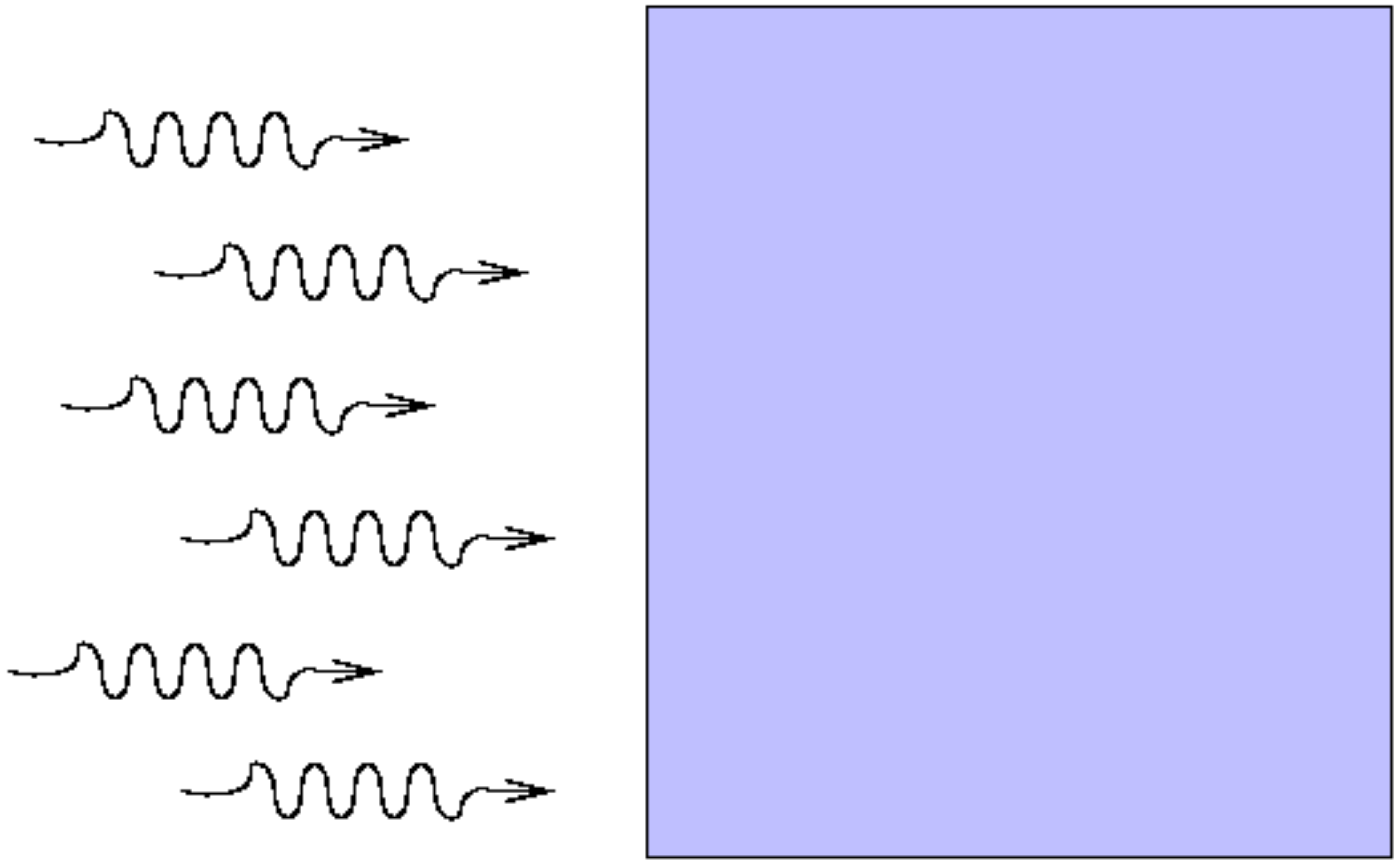
  which
  is the fundamental quantity
  for radiative processes



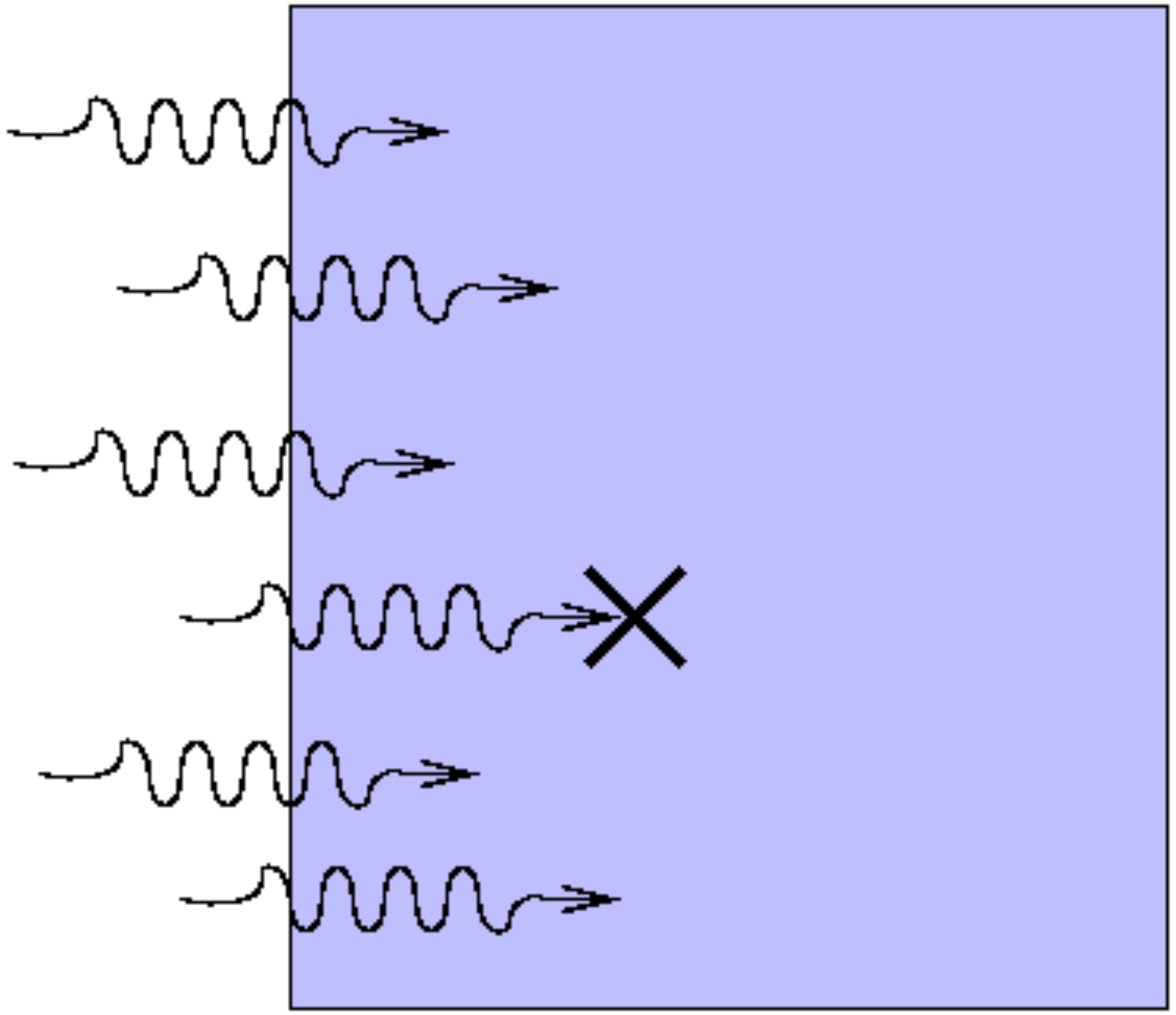
- Units:
  erg s^{-1} cm^{-2} sr^{-1} Hz^{-1}
- Astro-lingo: color is the difference between frequency bands

physical picture of optical depth

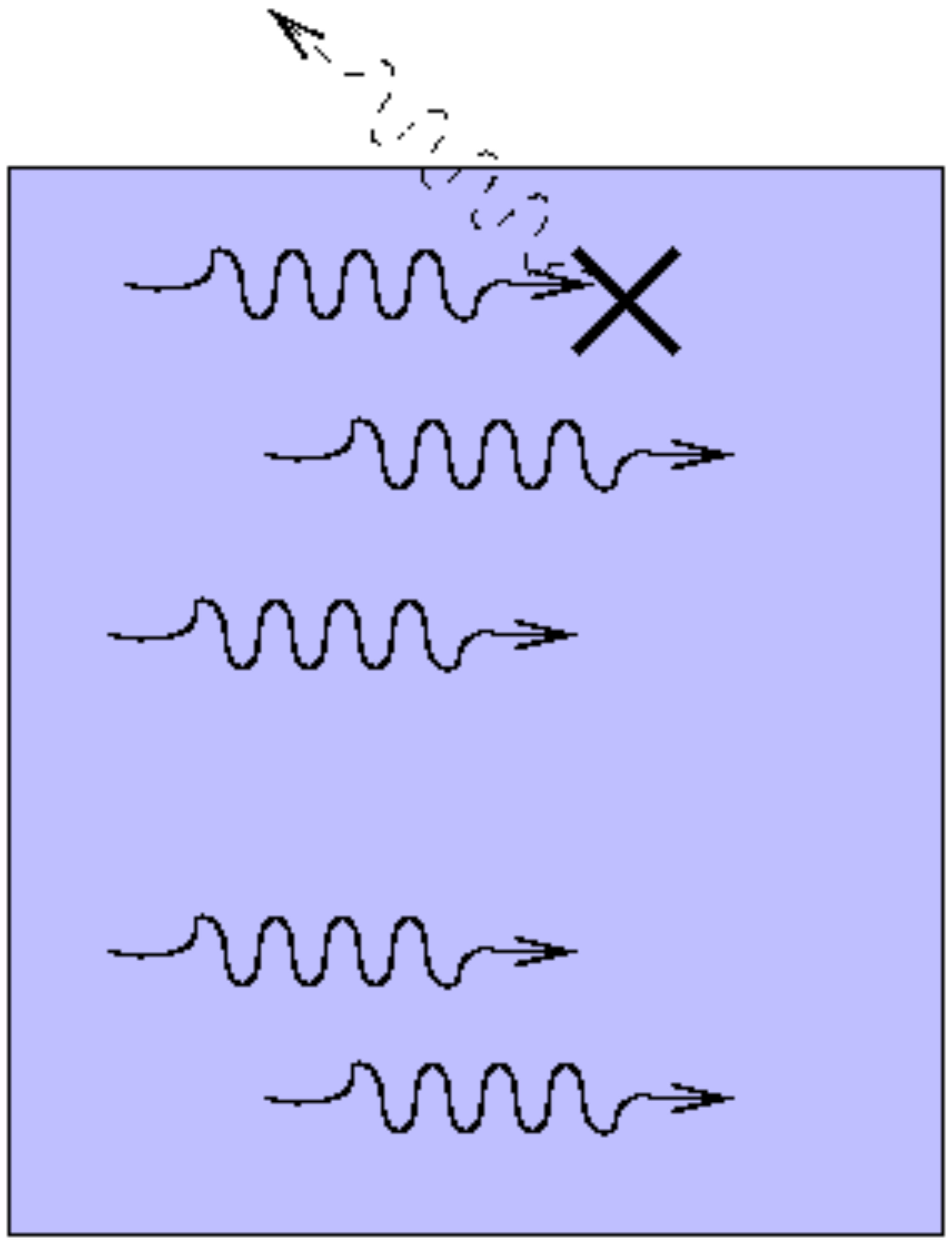
incoming rays



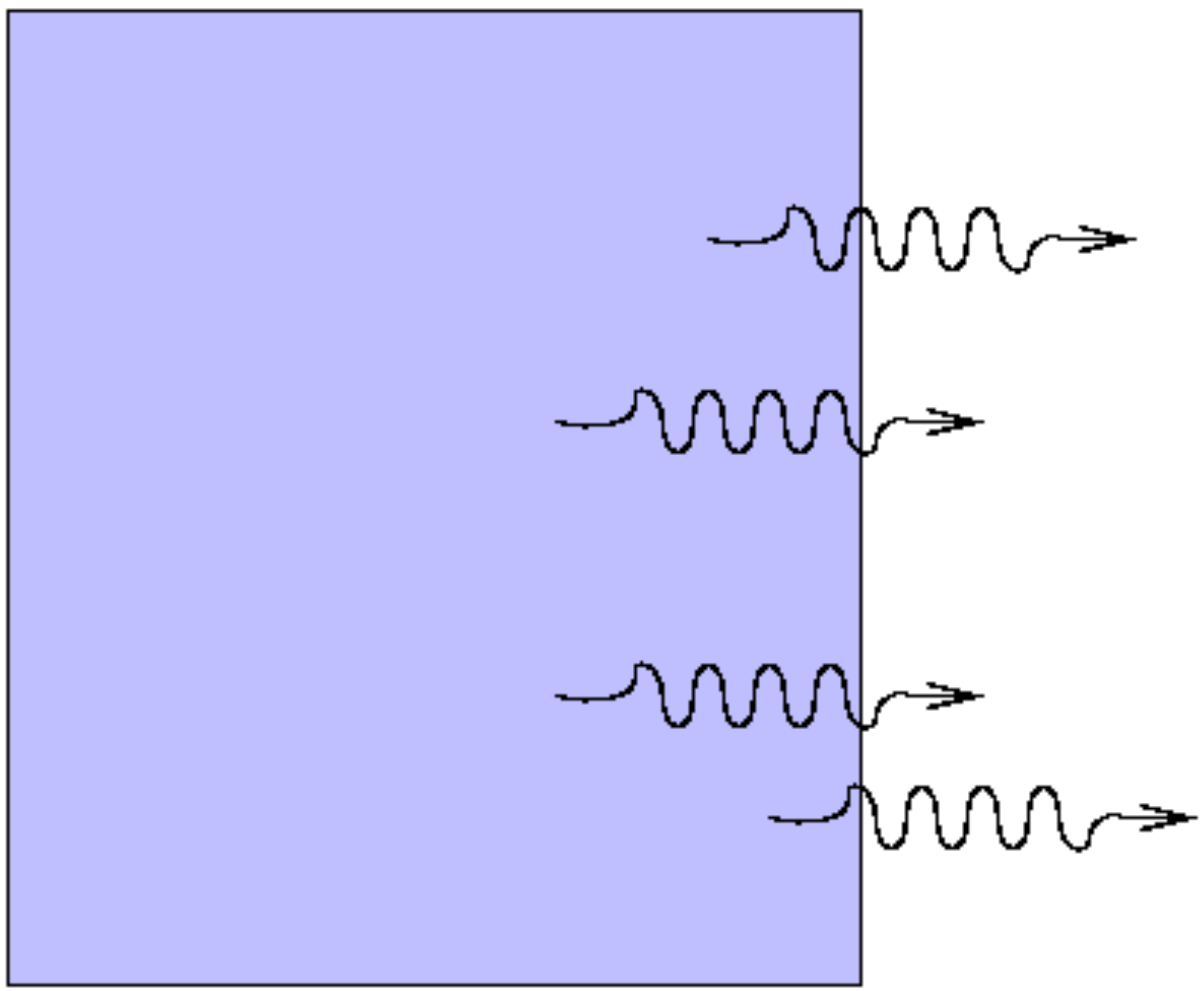
absorption



scattering



outgoing rays



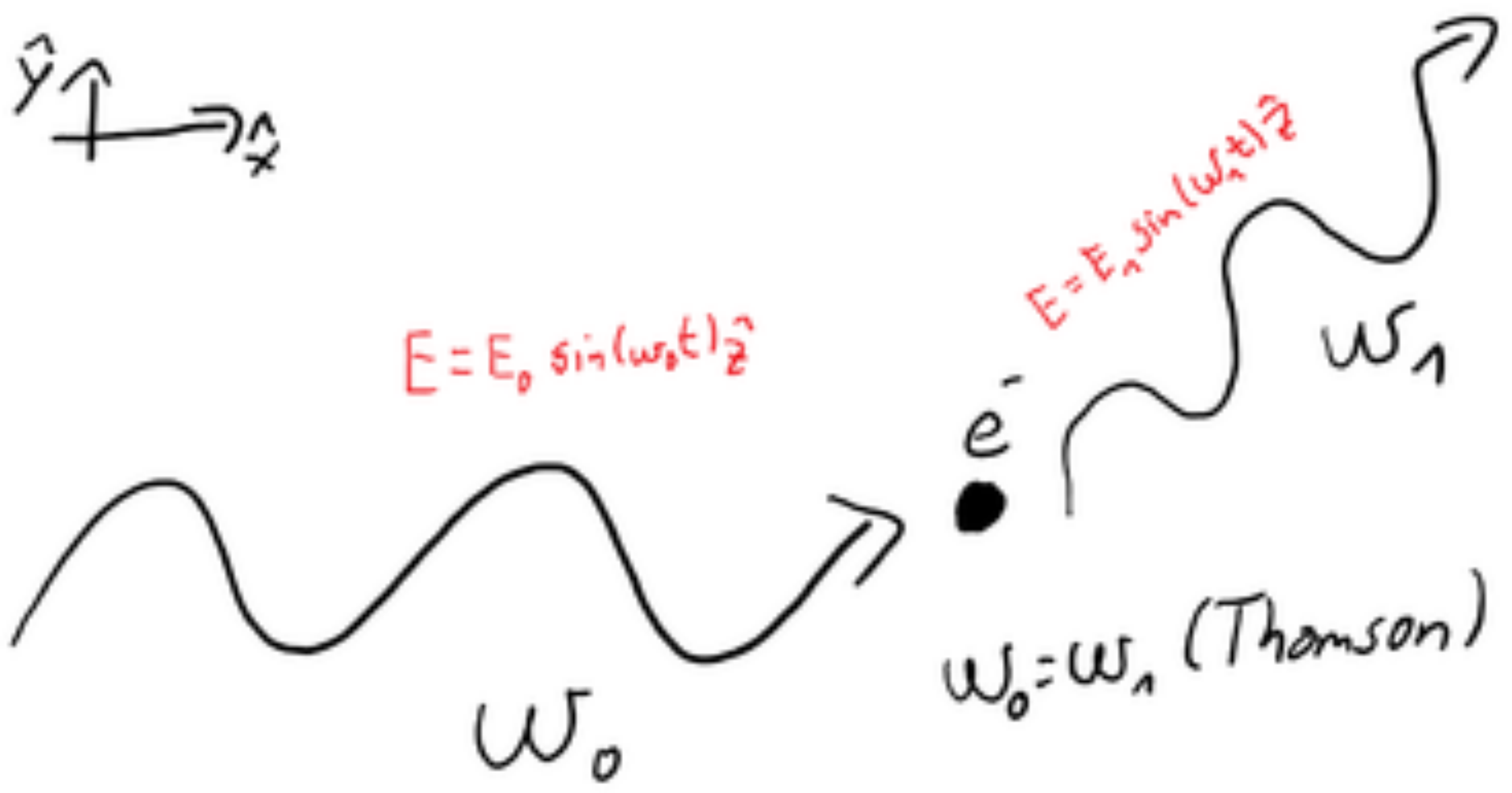
$h\nu \ll m_e c^2$

Thomson Scattering

Does not change the photon energy, just the direction

non-relativistic scattering

elastic scattering




volume element in phase space
scatter out
scatter in
time

- Temperature anisotropies then evolve as

$$\frac{d(\Theta+\psi)}{d\eta} - (\dot{\phi}+\dot{\psi}) = \underbrace{-an_e\sigma_T\Theta}_{\text{out-scattering}} + \underbrace{an_e\sigma_T \boldsymbol{e}\cdot\boldsymbol{v}_b}_{\text{Doppler}} + \underbrace{\frac{3an_e\sigma_T}{16\pi}\int d\hat{\boldsymbol{m}}\,\Theta(\hat{\boldsymbol{m}})[1+(\boldsymbol{e}\cdot\hat{\boldsymbol{m}})^2]}_{\text{in-scattering}}$$

Q: If Thomson Scattering process does not transfer energy, how to mix hot and cold region via TS?

A:

Scattering is a process that **does not** remove energy from the radiation field, but may redirect it.

NOTE: Scattering can be thought of as **absorption** of radiative energy followed by **re-emission** back to the electromagnetic field with negligible conversion of energy. Thus, scattering can remove radiative energy of a light beam traveling in one direction, but can be a "source" of radiative energy for the light beams traveling in other directions.

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}$$

T ↗ I_ν ↗ Number of photon in the volume element ↗

More photons are scattered in via TS, the hotter. And vice versa.

概率分布函数

$$f = \frac{1}{\exp(\frac{v}{T+\delta T})-1}$$

T ← t　　δT ← (t,$\underline{x}$)

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial \vec{x}}\frac{\partial \vec{x}}{\partial t}+\frac{\partial f}{\partial \vec{v}}\frac{\partial \vec{v}}{\partial t}=C$$

$\frac{\partial \vec{x}}{\partial t}$：速度　　$\frac{\partial \vec{v}}{\partial t}$：加速度 / 外力（引力）

要求解的量：　$\Theta(\vec{x},\hat{v},t)=\delta T/T$　温度密度比

$\vec{x}$：不均匀性　　$\hat{v}$：各向异性 **（不依赖于动量的绝对值）？**

对$\underline{x}$进行Fourier变换，因为偏微分方程是很难求解的

$$\Theta(k,\hat{k},\hat{v},t)$$

CMB温度的多级矩 / multipoles

$$\Theta(k,\mu,t) \qquad \mu = \hat{k} \bullet \hat{v}$$

$$\Theta_\ell(k,t) = \int \Theta(k,\mu,t) P_l(\mu) d\mu$$

CMB光子温度涨落是一个高斯性的随机变量，
该系统可以**完全**由其**两点相关函数**来刻画。

这里，我们用其**球谐空间 / spherical harmonics**中，
的两点相关函数，角度功率谱 / angular power spectrum

$$\left\langle \frac{\delta T}{T} \frac{\delta T}{T} \right\rangle \rightarrow C_\ell$$

计算和测量C_ell是CMB物理的**中心任务**！！！

玻尔兹曼方程/Boltzmann hierarchy

$$\dot{\delta}_\gamma = -\frac{4}{3}\theta_\gamma - \frac{2}{3}\dot{h}\,,$$

(ell=0, 能量密度/单极距)

$$\dot{\theta}_\gamma = k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + an_e\sigma_T(\theta_b - \theta_\gamma)\,,$$

(ell=1, 速度 / 偶极距)

$$\dot{F}_{\gamma\,2} = 2\dot{\sigma}_\gamma = \frac{8}{15}\theta_\gamma - \frac{3}{5}kF_{\gamma\,3} + \frac{4}{15}\dot{h} + \frac{8}{5}\dot{\eta} - \frac{9}{5}an_e\sigma_T\sigma_\gamma + \frac{1}{10}an_e\sigma_T\left(G_{\gamma\,0} + G_{\gamma\,2}\right)$$

(ell=2, 剪切粘滞 / 四极距)

$$\dot{F}_{\gamma\,l} = \frac{k}{2l+1}\left[lF_{\gamma\,(l-1)} - (l+1)F_{\gamma\,(l+1)}\right] - an_e\sigma_T F_{\gamma\,l}\,, \quad l \geq 3\,,$$

$$\dot{G}_{\gamma\,l} = \frac{k}{2l+1}\left[lG_{\gamma\,(l-1)} - (l+1)G_{\gamma\,(l+1)}\right] + an_e\sigma_T\left[-G_{\gamma\,l} + \frac{1}{2}\left(F_{\gamma\,2} + G_{\gamma\,0} + G_{\gamma\,2}\right)\left(\delta_{l0} + \frac{\delta_{l2}}{5}\right)\right]$$

耦合的线性常微分方程组，（方程数目**巨大**！）

数值实现起来，**十分费时**！

CMBFast软件包

成功地将，程序运行时间降为几分钟

Seljak

Zaldarriaga

orders of magnitude reduction in CPU time when compared to standard methods and typically requires a few minutes on a workstation for a single model. The method should be especially useful for accurate determinations of

Seljak et. al. 96'

关键技巧：将与宇宙学模型无关的，
纯几何的球谐Bessel函数，分离，提前计算出来。

真正与模型相关的方程，数目只有**30个**左右。

现在，CMBFast已停止维护。代替其的，
是有Lewis开发的**CAMB**（CAMBridge）Lewis 99’

http://camb.info　运行时间大约为1s

CAMB基于Fortran

Lewis

除此之外，还有**CLASS**，Lesgourgues 11’

CLASS基于C

Lesgourgues

CAMB开发时间更久，被测试得更为全面；
CLASS有许多算法上的优化，例如：中微子部分处理得更好

Further reading

CMB basics

- Wayne Hu's excellent website (http://background.uchicago.edu/~whu/)

- Hu & White's "Polarization primer" (arXiv:astro-ph/9706147)

- AC's summer school lecture notes (arXiv:0903.5158 and arXiv:astro-ph/0403344)

https://cosmology.unige.ch/sites/default/files/media/Anthony_Challinor_CMB_lectures_jun13.pdf

CMB lensing

- Lewis & AC's "Weak gravitational lensing of the CMB" (arXiv:astro-ph/0601594)

Textbook

- *Morden Cosmology* by Dodelson

- *The Cosmic Microwave Background* by Ruth Durrer