CMB physics

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1. 相关知识准备

1.1 高斯统计

Key concept

- Phase info v.s. Power spectrum
- Different points in real space are correlated
- Different k-modes in Fourier space are uncorrelated
- White noise

Cosmic density field

For a given cosmology, the density field at a cosmic time t, is described by

$$\delta(\mathbf{x},t) \quad \text{or} \quad \delta_{\mathbf{k}}(t).$$

How to specify a linear density field? to specify $\delta(\mathbf{x})$ for all $\mathbf{x}$ or to specify $\delta_{\mathbf{k}}$ for all $\mathbf{k}$? NO!

- We consider the cosmic density field to be the realization of a random process, which is descibed by a probability distribution function:

$$\mathcal{P}_x(\delta_1, \delta_2, \cdots, \delta_N)\, \mathrm{d}\delta_1\, \mathrm{d}\delta_2 \cdots \mathrm{d}\delta_N, \qquad (N \to \infty)$$

Thus, we emphsize the properties of $\mathcal{P}_x$, rather than the exact form of $\delta(\mathbf{x})$.

[from Houjun Mo]

- The form of $\mathcal{P}_x(\delta_1, \delta_2, \cdots, \delta_N)$: is determined if we know all of its moments:

$$\left\langle \delta_1^{\ell_1} \delta_2^{\ell_2} \cdots \delta_N^{\ell_N} \right\rangle \equiv \int \delta_1^{\ell_1} \delta_2^{\ell_2} \cdots \delta_N^{\ell_N} \mathcal{P}_x(\delta_1, \delta_2, \cdots, \delta_N)\, \mathrm{d}\delta_1\, \mathrm{d}\delta_2 \cdots \mathrm{d}\delta_N,$$

where $(\ell_1, \ell_2, \cdots, \ell_N) = 0, 1, 2, \cdots$.

In real space:

$$\langle \delta(\mathbf{x}) \rangle = 0, \quad \xi(x) = \langle \delta_i \delta_j \rangle, \quad \text{where} \quad x \equiv |\mathbf{x}_i - \mathbf{x}_j|.$$

In Fourier space:

$$\langle \delta_{\mathbf{k}} \rangle = 0, \quad P(k) \equiv V_{\mathrm{u}} \left\langle |\delta_{\mathbf{k}}|^2 \right\rangle \equiv V_{\mathrm{u}} \langle \delta_{\mathbf{k}} \delta_{-\mathbf{k}} \rangle = \int \xi(\mathbf{x}) \exp(i\mathbf{k} \cdot \mathbf{x})\, \mathrm{d}^3\mathbf{x},$$

In general, it is quite difficult to describe a random field.

[from Houjun Mo]

Gaussian Random Fields

- In real space:

$$\mathcal{P}(\delta_1,\delta_2,\cdots,\delta_n) = \frac{\exp(-Q)}{[(2\pi)^n \det(\mathcal{M})]^{1/2}}; \quad Q \equiv \frac{1}{2}\sum_{i,j}\delta_i\left(\mathcal{M}^{-1}\right)_{ij}\delta_j,$$

where $\mathcal{M}_{ij} \equiv \langle\delta_i\delta_j\rangle$. For a homogeneous and isotropic field, all the multivariate distribution functions are invariant under spatial translation and rotation, and so are completely determined by the two-point correlation function $\xi(x)$!

[from Houjun Mo]

- In Fourier space:

$$\delta_{\mathbf{k}} = A_{\mathbf{k}} + iB_{\mathbf{k}} = |\delta_{\mathbf{k}}| \exp(i\varphi_{\mathbf{k}}).$$

Since $\delta(\mathbf{x})$ is real, we have $A_{\mathbf{k}} = A_{-\mathbf{k}}$, $B_{\mathbf{k}} = -B_{-\mathbf{k}}$, and so we need only Fourier modes with $\mathbf{k}$ in the upper half space to specify $\delta(\mathbf{x})$. It is then easy to prove that, for $\mathbf{k}$ in the upper half space,

$$\langle A_{\mathbf{k}} A_{\mathbf{k}'} \rangle = \langle B_{\mathbf{k}} B_{\mathbf{k}'} \rangle = \frac{1}{2} V_{\mathrm{u}}^{-1} P(k) \delta^{(\mathrm{D})}_{\mathbf{k}\mathbf{k}'}; \quad \langle A_{\mathbf{k}} B_{\mathbf{k}'} \rangle = 0,$$

Thus As a result, the multivariate distribution functions of $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are factorized according to $\mathbf{k}$, each factor being a Gaussian:

$$\mathcal{P}(\alpha_{\mathbf{k}})\,\mathrm{d}\alpha_{\mathbf{k}} = \frac{1}{[\pi V_{\mathrm{u}}^{-1} P(k)]^{1/2}} \exp\left[-\frac{\alpha_{\mathbf{k}}^2}{V_{\mathrm{u}}^{-1} P(k)}\right] \mathrm{d}\alpha_{\mathbf{k}},$$

[from Houjun Mo]

In terms of $|\delta_{\mathbf{k}}|$ and $\varphi_{\mathbf{k}}$, the distribution function for each mode, $\mathcal{P}(A_{\mathbf{k}})\mathcal{P}(B_{\mathbf{k}})\,\mathrm{d}A_{\mathbf{k}}\,\mathrm{d}B_{\mathbf{k}}$, can be written as

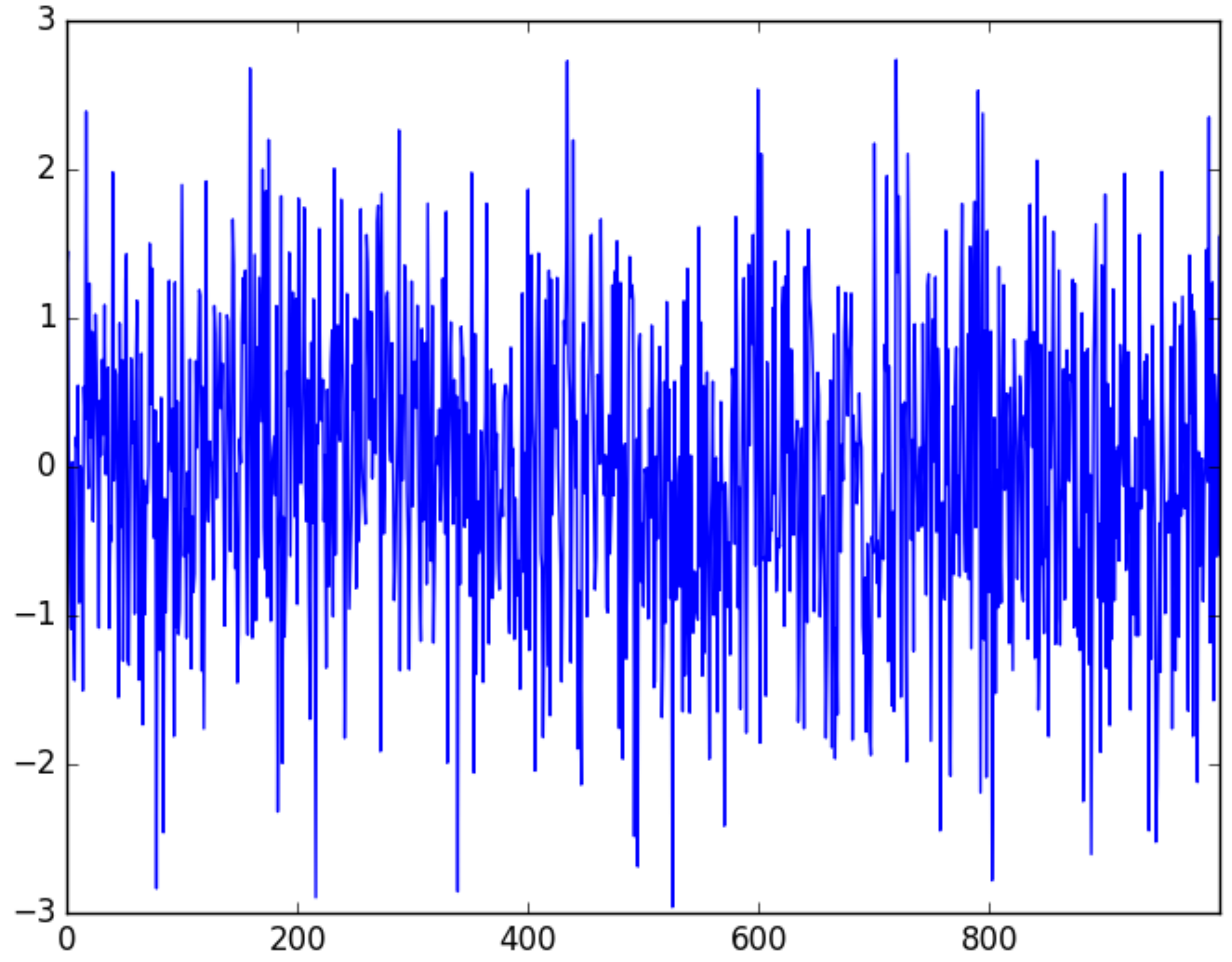
$$\mathcal{P}(|\delta_{\mathbf{k}}|,\varphi_{\mathbf{k}})\,\mathrm{d}|\delta_{\mathbf{k}}|\,\mathrm{d}\varphi_{\mathbf{k}} = \exp\left[-\frac{|\delta_{\mathbf{k}}|^2}{2V_{\mathrm{u}}^{-1}P(k)}\right]\frac{|\delta_{\mathbf{k}}|\,\mathrm{d}|\delta_{\mathbf{k}}|}{V_{\mathrm{u}}^{-1}P(k)}\frac{\mathrm{d}\varphi_{\mathbf{k}}}{2\pi}.$$

Thus, for a Gaussian field, different Fourier modes are mutually independent, so are the real and imaginary parts of individual modes. This, in turn, implies that the phases $\varphi_{\mathbf{k}}$ of different modes are mutually independent and have random distribution over the interval between 0 and 2π.

$P(k)$ is the only function we need!

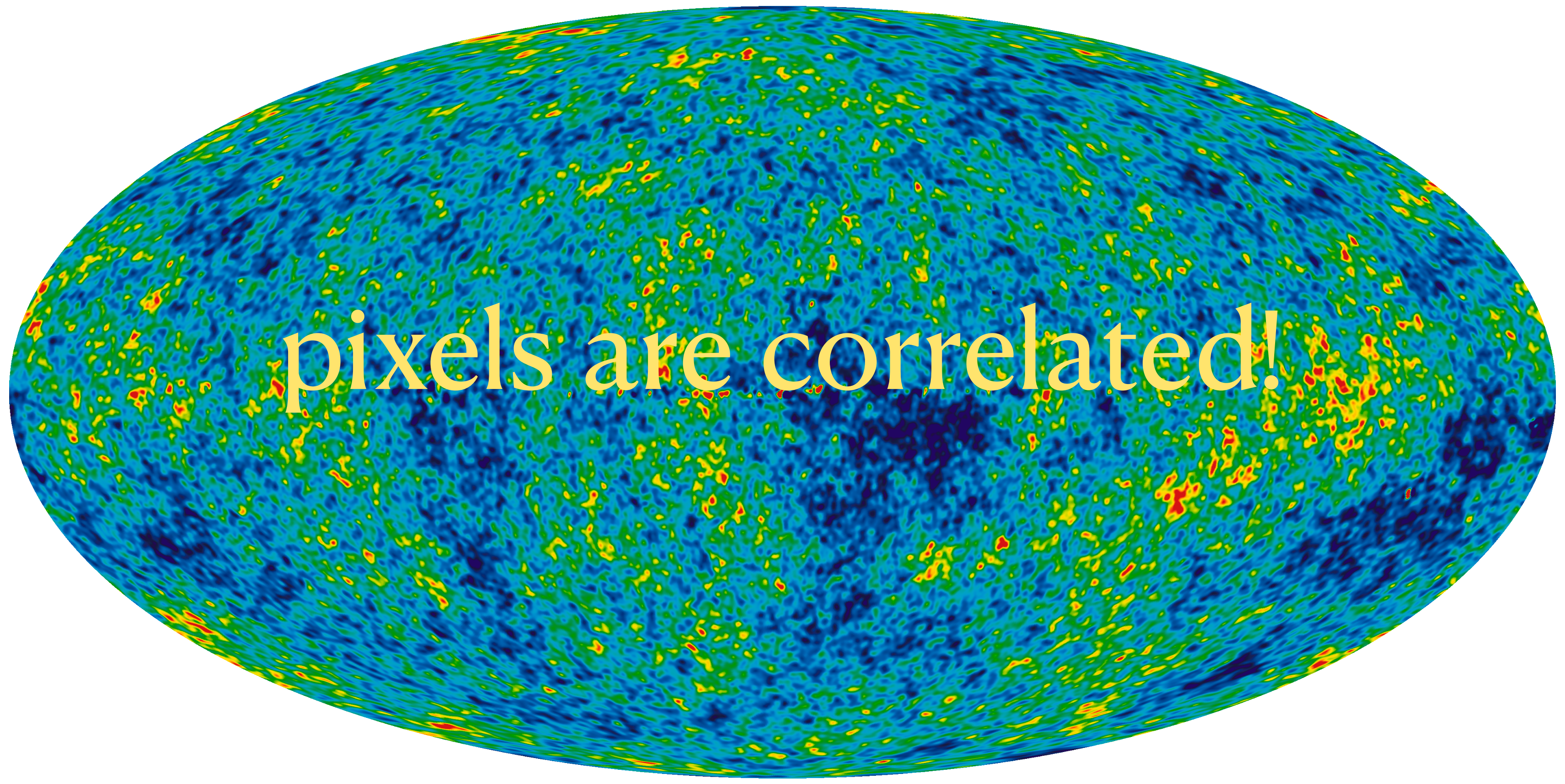
φ_{k} **: is uniformly distributed between 0 and 2pi**

[from Houjun Mo]

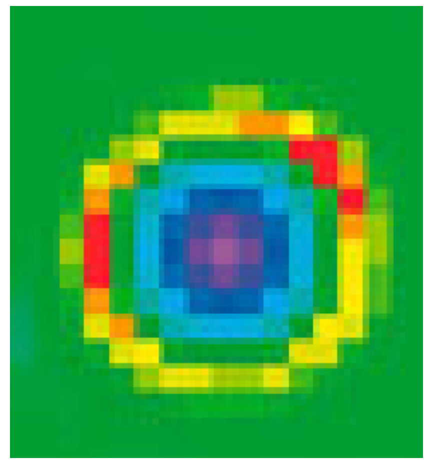


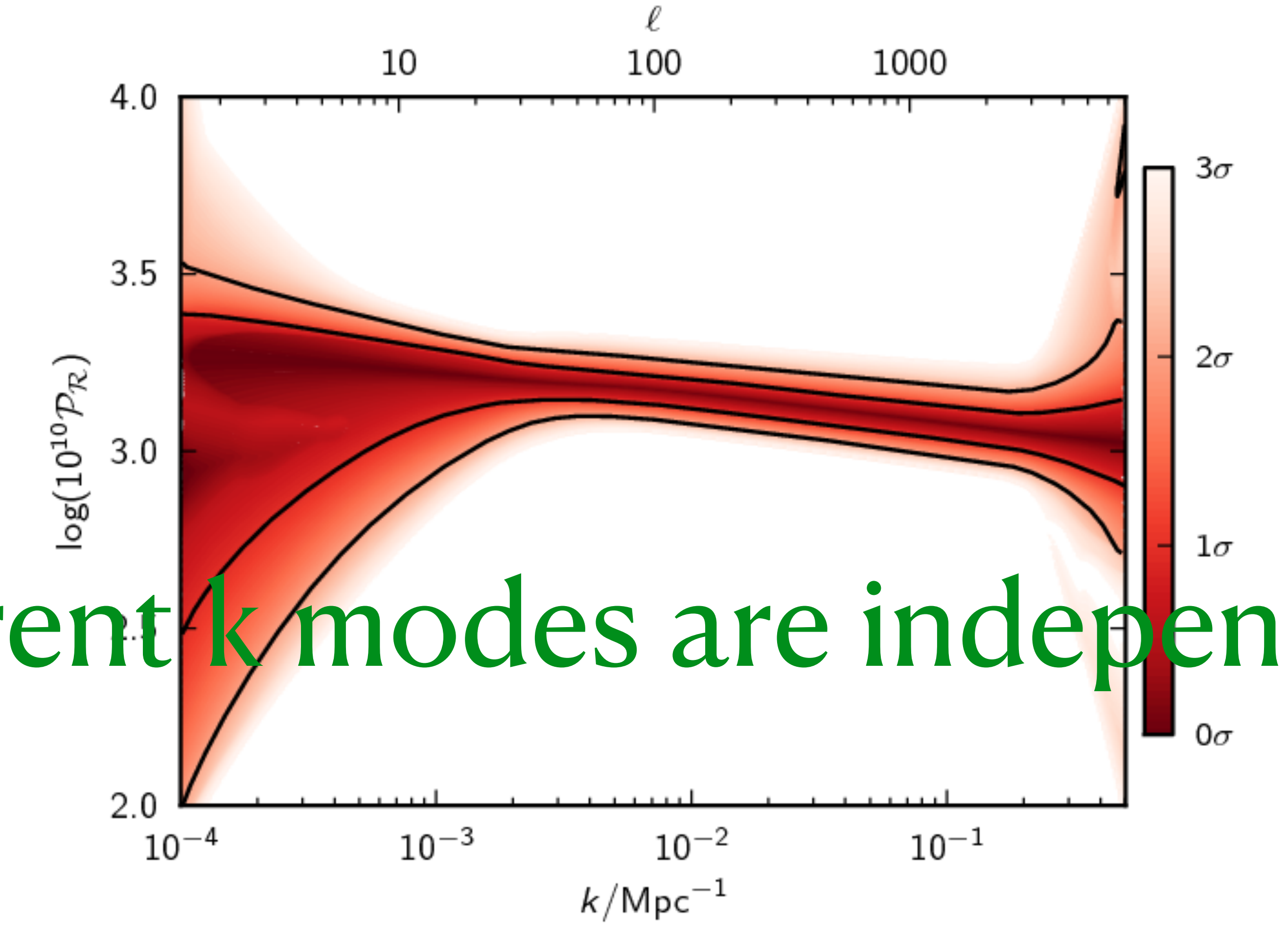
White Noise

- time domain: delta(t); different time is independent
- frequency domain: constant spectrum (equal weight from each frequencies)
- For any other type power spectrum, the data in the real/time domain, are correlated with some length.



pixels are correlated!





different k modes are independent!

Although power spectrum can NOT tell us ALL the statistics, still it is informative

real gauss random field

$$\hat{s}(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \hat{s}_{\vec{k}}$$

complex gauss random field

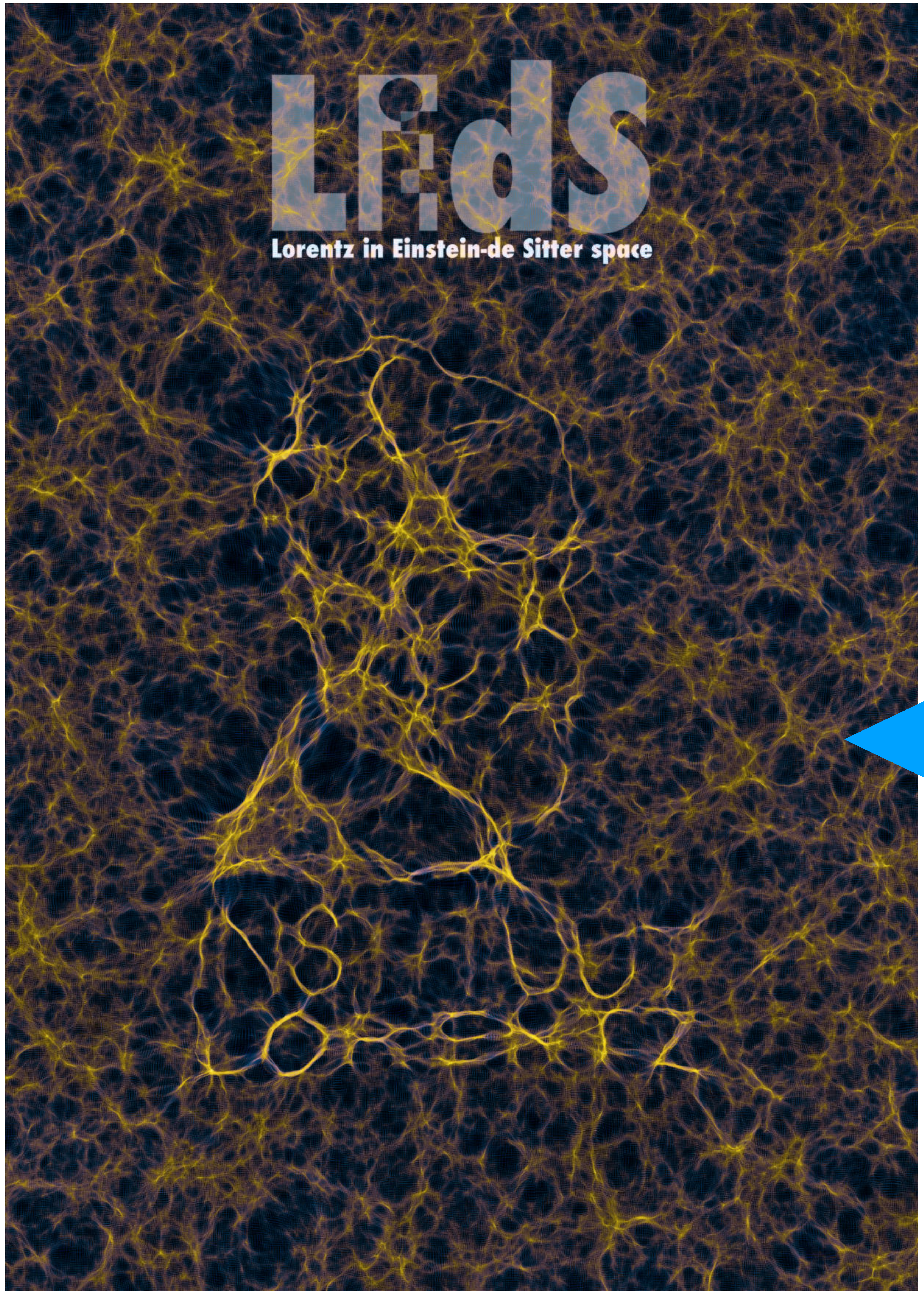
$$= \lim_{L\to\infty} \sum_{\vec{n}=-\infty}^{\infty} L^{-3} e^{i\frac{2\pi\vec{n}}{L}\cdot\vec{x}} \hat{s}_{\frac{2\pi\vec{n}}{L}},$$

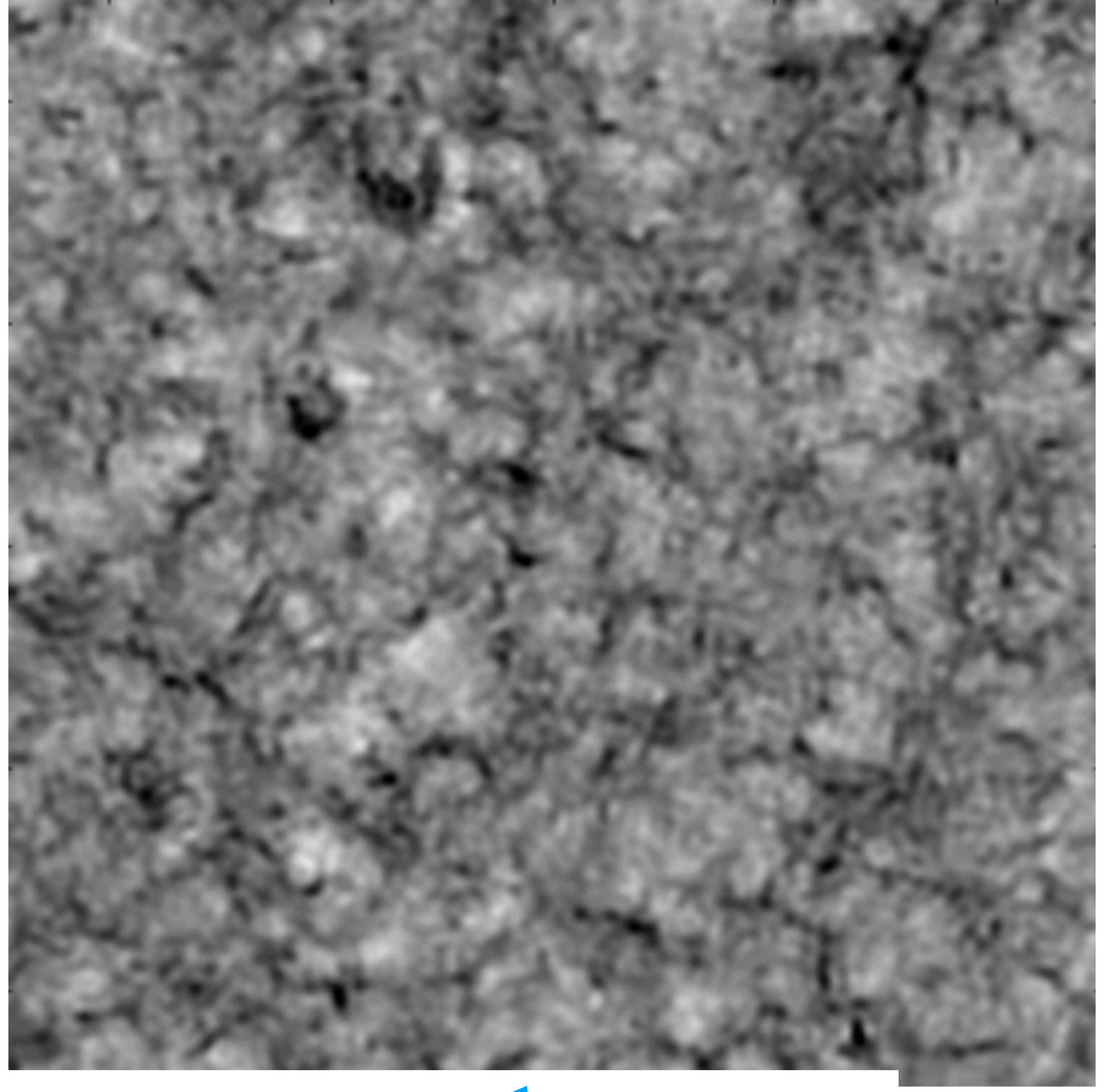
$$\left\langle \hat{s}_{\frac{2\pi\vec{n}}{L}} \hat{s}^*_{\frac{2\pi\vec{n}'}{L}} \right\rangle = L^{-3}\delta_{\vec{n},\vec{m}} \underline{P_{\hat{s}}\left(\left|\frac{2\pi\vec{n}}{L}\right|\right)}$$

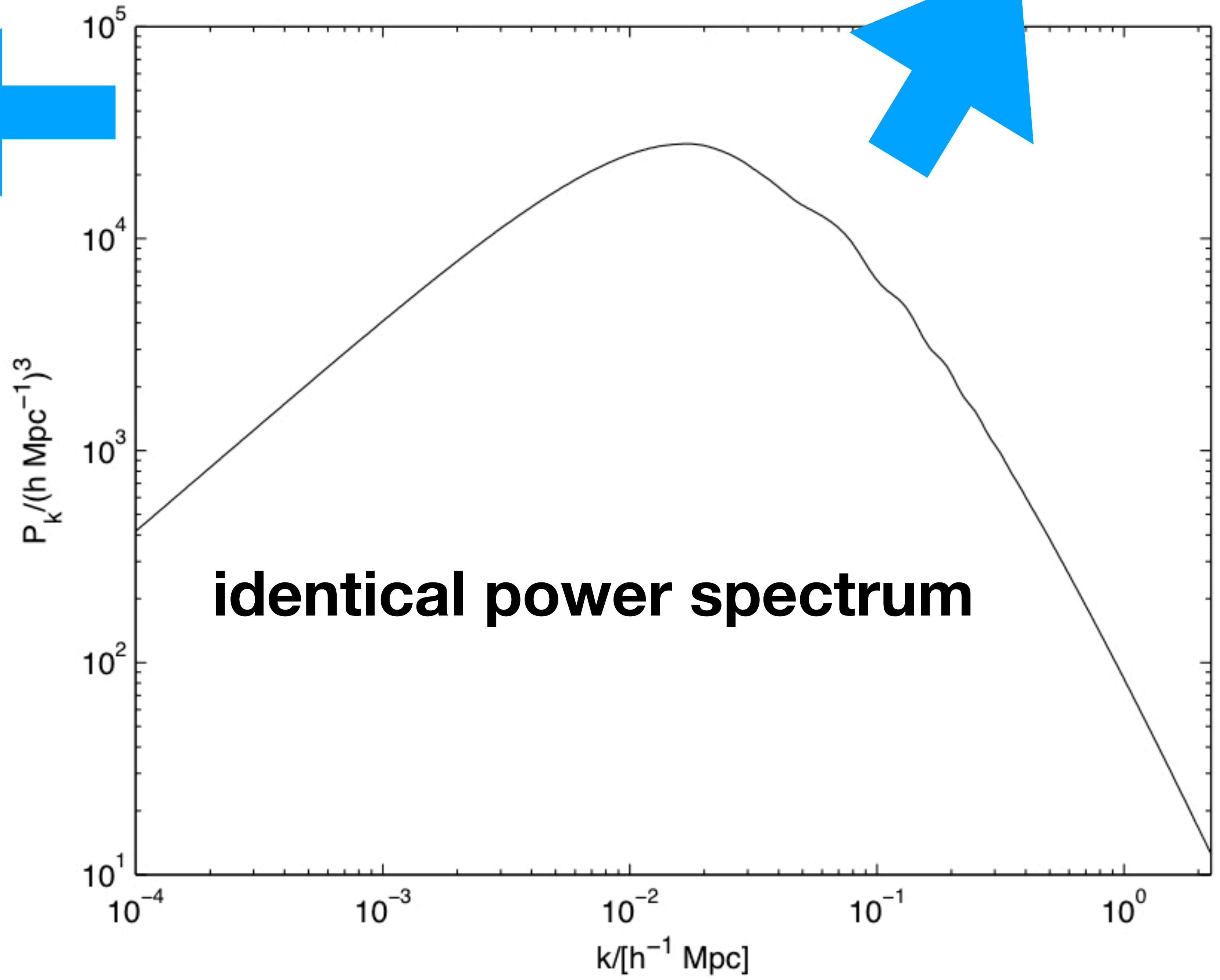
power spectrum only give us the info encoded in Amplitude

$$\hat{s}(\vec{k}) \sim \hat{A}(\vec{k})\cancel{e^{i\hat{\phi}(\vec{k})}}$$

Loss info encoded in the phase!


Lorentz in Einstein-de Sitter space



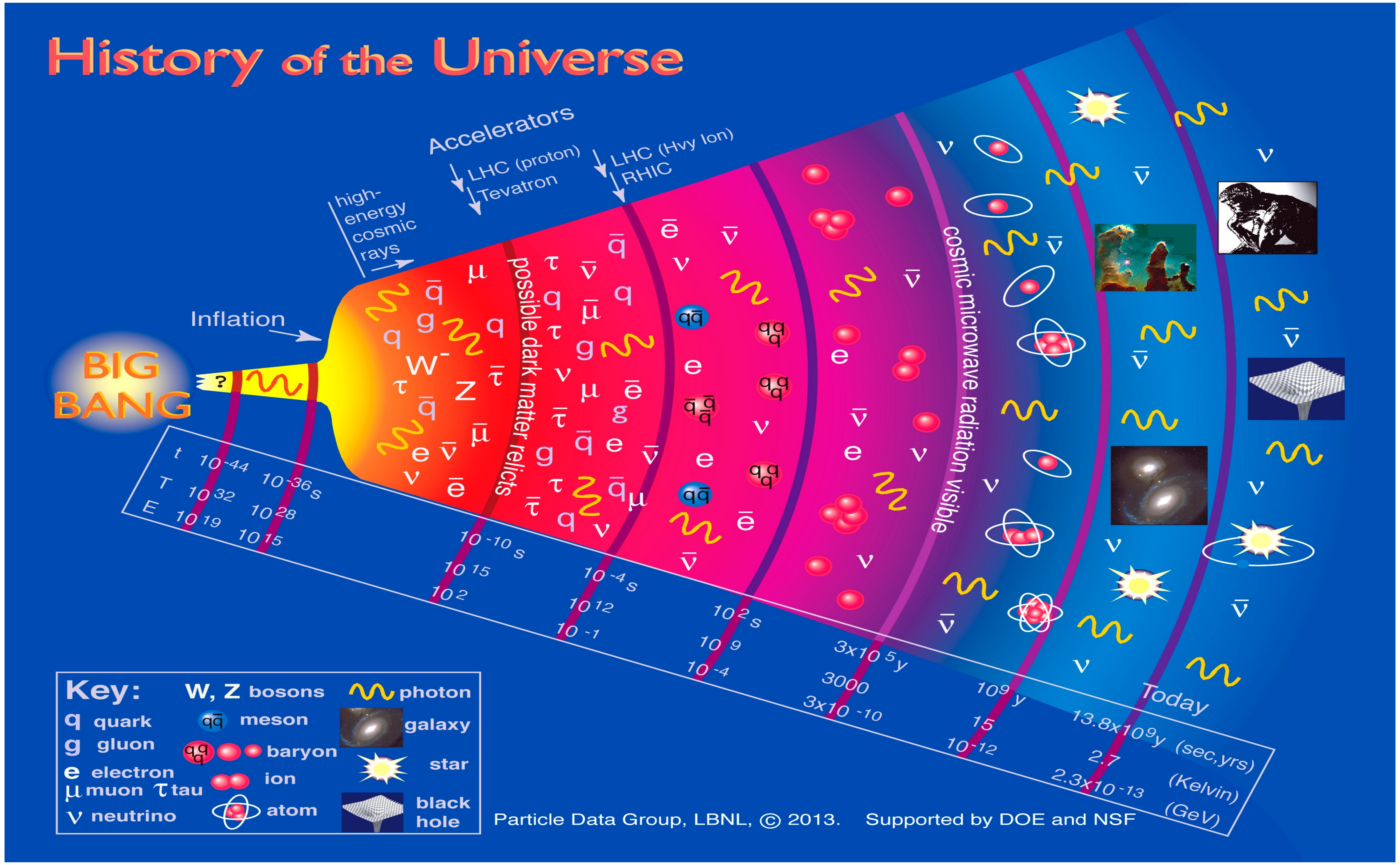

identical power spectrum
$P_k/(h\ \mathrm{Mpc}^{-1})^3$
$k/[h^{-1}\ \mathrm{Mpc}]$
10^{5}
10^{4}
10^{3}
10^{2}
10^{1}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}

1. 相关知识准备

Key concept

1.2 Primordial Power spectrum

- Quantum original
- Nearly massless inflaton (slow roll parameter)
- Scalar perturbation does not directly measure the inflation energy scale, tensor does.
- parametric form of primordial power spectrum



T~1GeV

quark -> proton&neutron

T~1MeV

nucleosynthesis: create light element D, He, Li

T~1eV

recombination: $p^+ + e^- \rightarrow H$

z~(100,1000)

dark ages

z~(6,10)

reionization

z<2

galaxy cluster

z<1

Dark energy

hot ⟶ cold

GR is a classical theory, does not involve any quantum phenomenon (no $\hbar$)

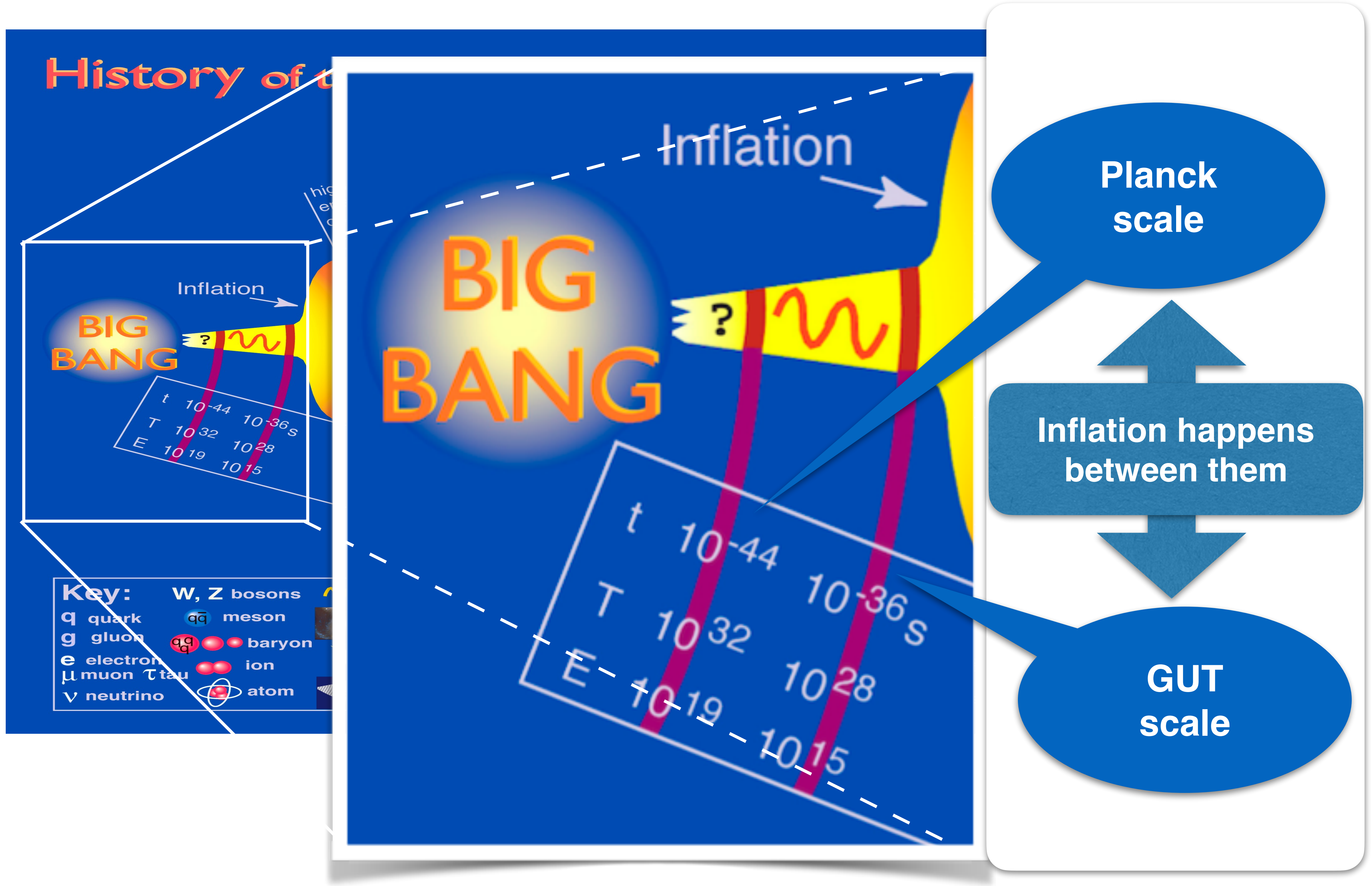
A typical Schwarzschild black hole radius: $\dfrac{2GM}{c^2}$

$$G_{\mu\nu}(x) + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}(x)$$

uncertainty principle: $\delta P \bullet \delta\lambda \sim \hbar$ the inertial energy of particle with mass M: $E = Mc^2$

Planck Mass $M_* \sim \sqrt{\hbar / G} \sim 10^{19} GeV$

when the system energy approaches Planck mass, we need to quantise gravity!

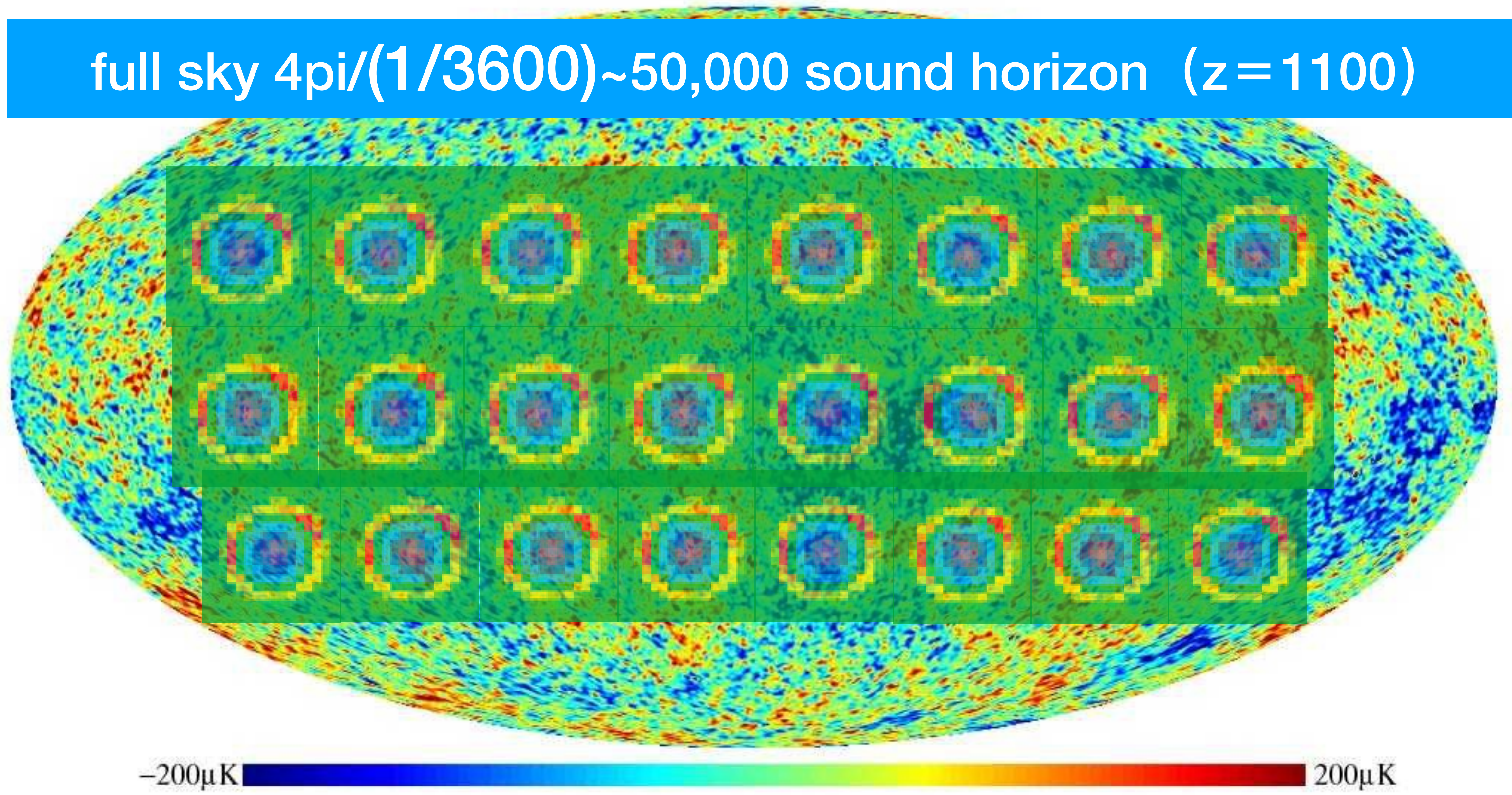


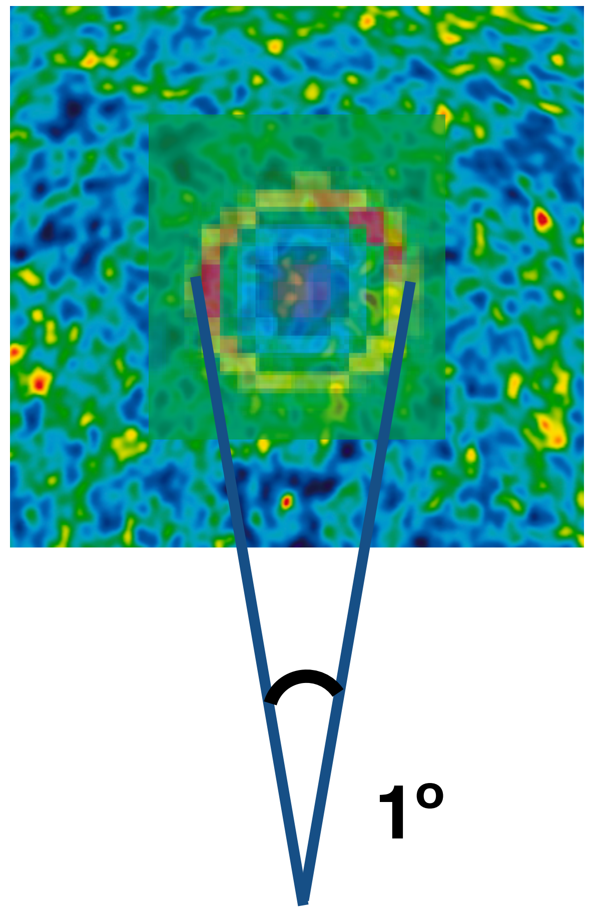
From t=0 to 10^{-44} s (Planck time), cosmic energy scale is above 10^{19} GeV (Planck energy)

why do we need inflation?

$1 deg^2 \sim (pi/180)^2 \sim 1/3600$

full sky 4pi/(1/3600)~50,000 sound horizon (z=1100)





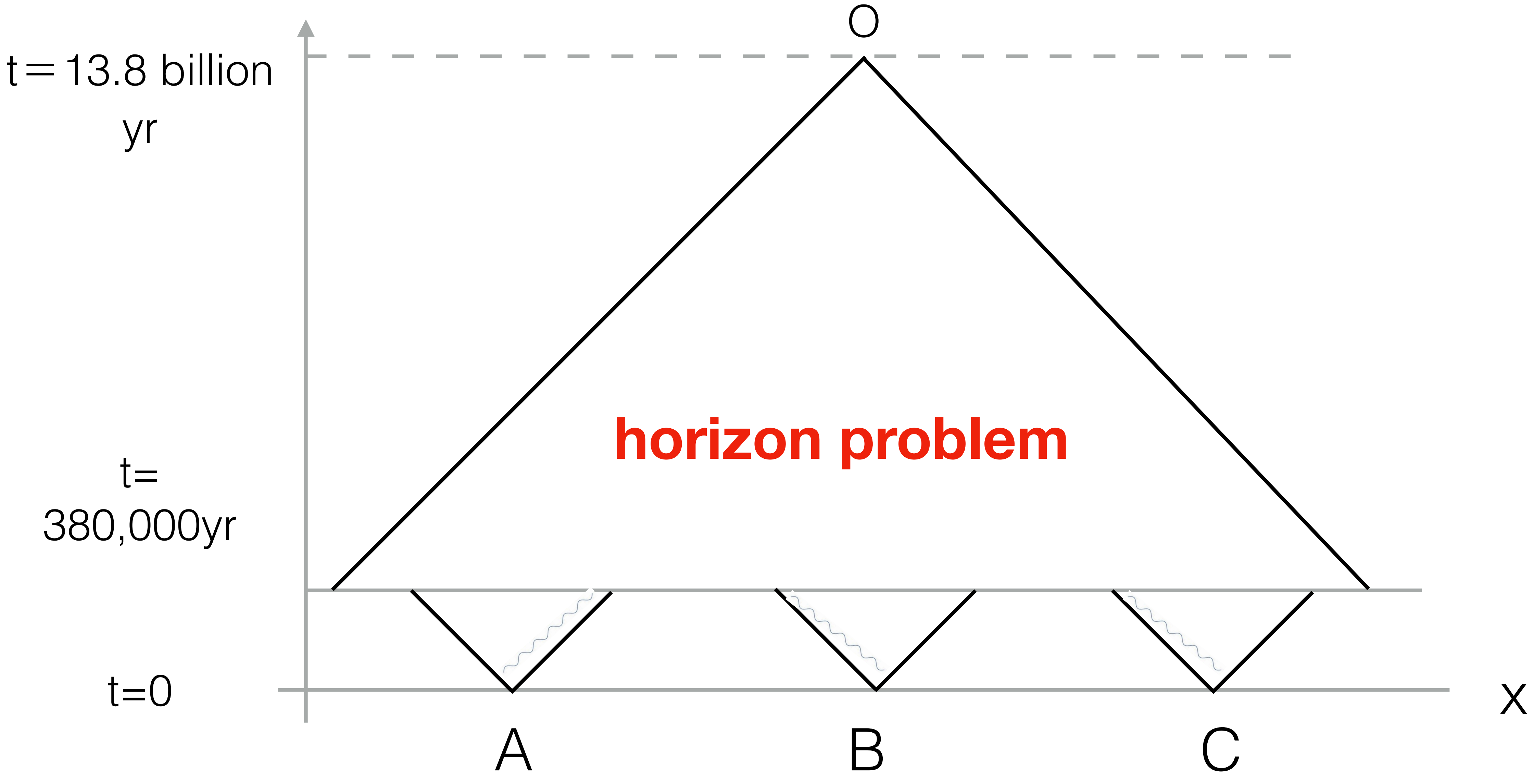
A photon from t=0, with velocity c/3, via 380,000yr can travel:
38×10^4 / 3 lyr ~3×10^4 pc

A photon from t=0, with velocity c, via 13.8 billion yr, can travel:
138×10^8 lyr~5×10^9 pc

remove the co-moving factor
$a_{z=0}/a_{z=1100} \sim 1000$

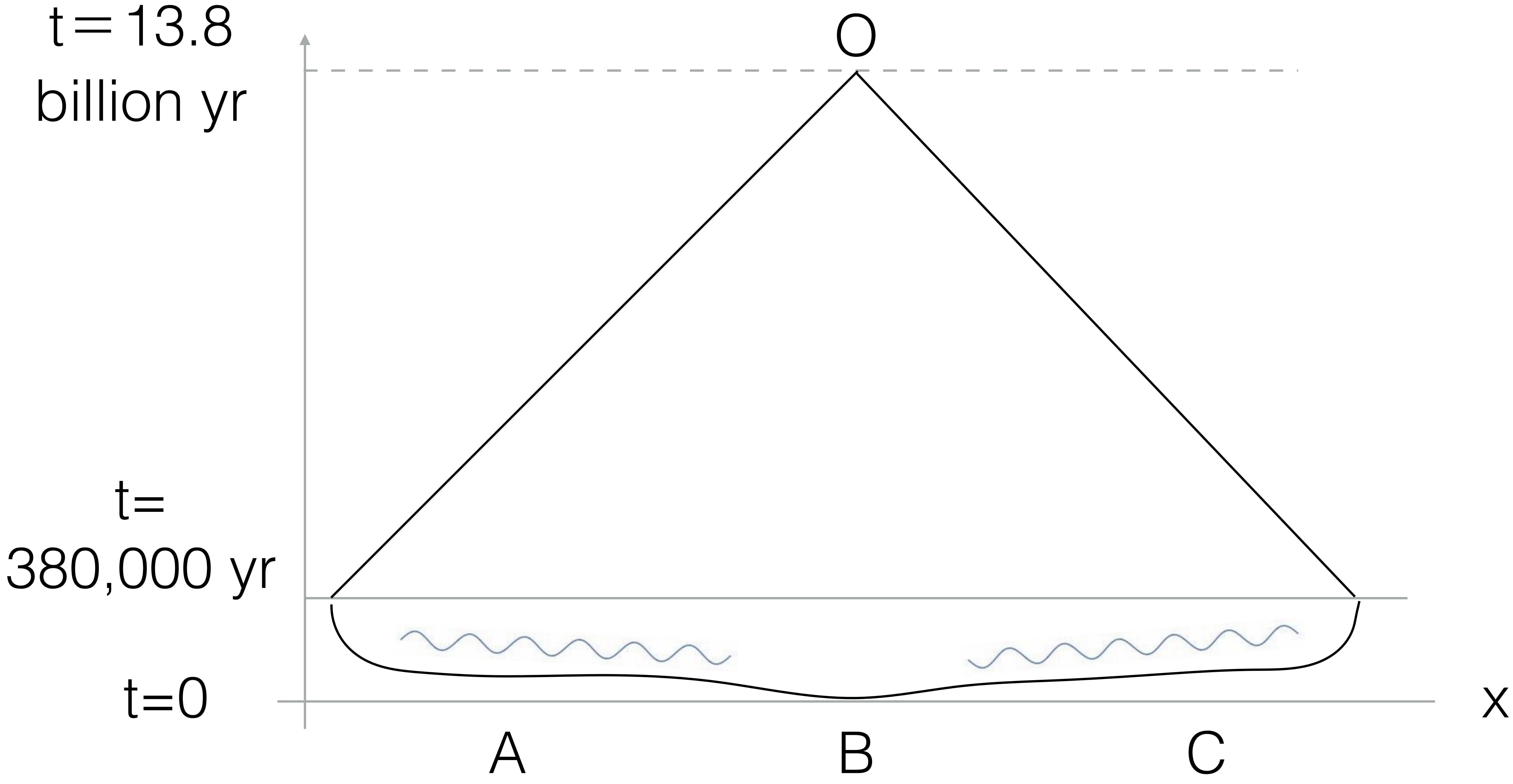
ratio: 5×10^9 / 3×10^4 / 10^3~140

2d sphere, totally 140^2~20,000 causal disconnected region

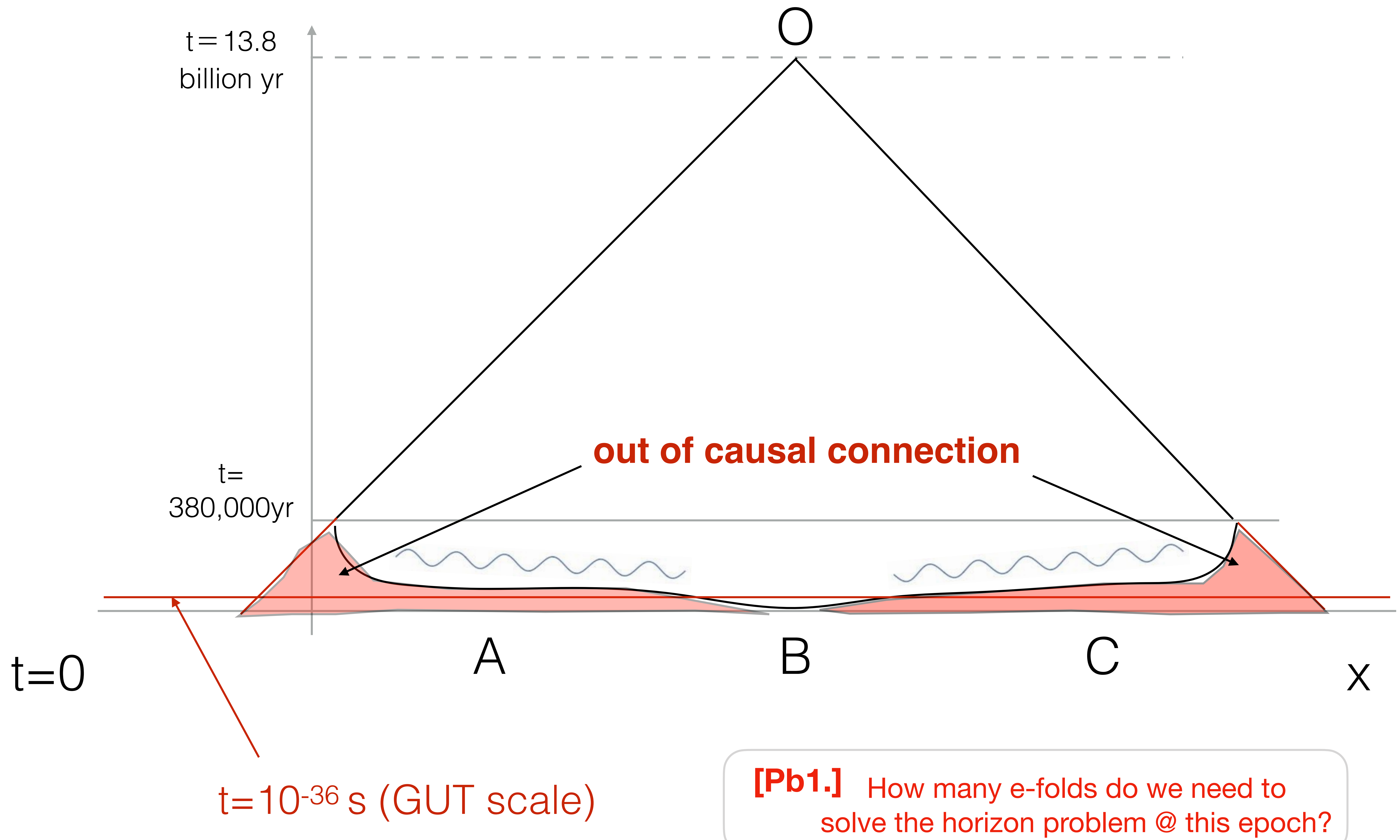


To solve horizon problem @z=1000, need enlarge the physical size of forward light-cone, by a factor 100.

$e^N \sim 100$, $N \sim 5$ (e-folding number)



continue to push back to GUT scale



[Pb1.] How many e-folds do we need to solve the horizon problem @ this epoch?

flatness problem

$$\boxed{H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}}$$

$\Omega = \rho/3M_{pl}^2 H^2$

$\Omega - 1 = \frac{k}{a^2 H^2}$

10^{60}

$|\Omega_k| < 0.005$

$10^{19}\,GeV$	$10^{-3}\,eV$	$10\ eV$
Planck era	DE era	equality era

$$\frac{\rho_{pl}}{\rho_{de}} = \left(\frac{E_{pl}}{E_{de}}\right)^4 \sim 10^{124}$$

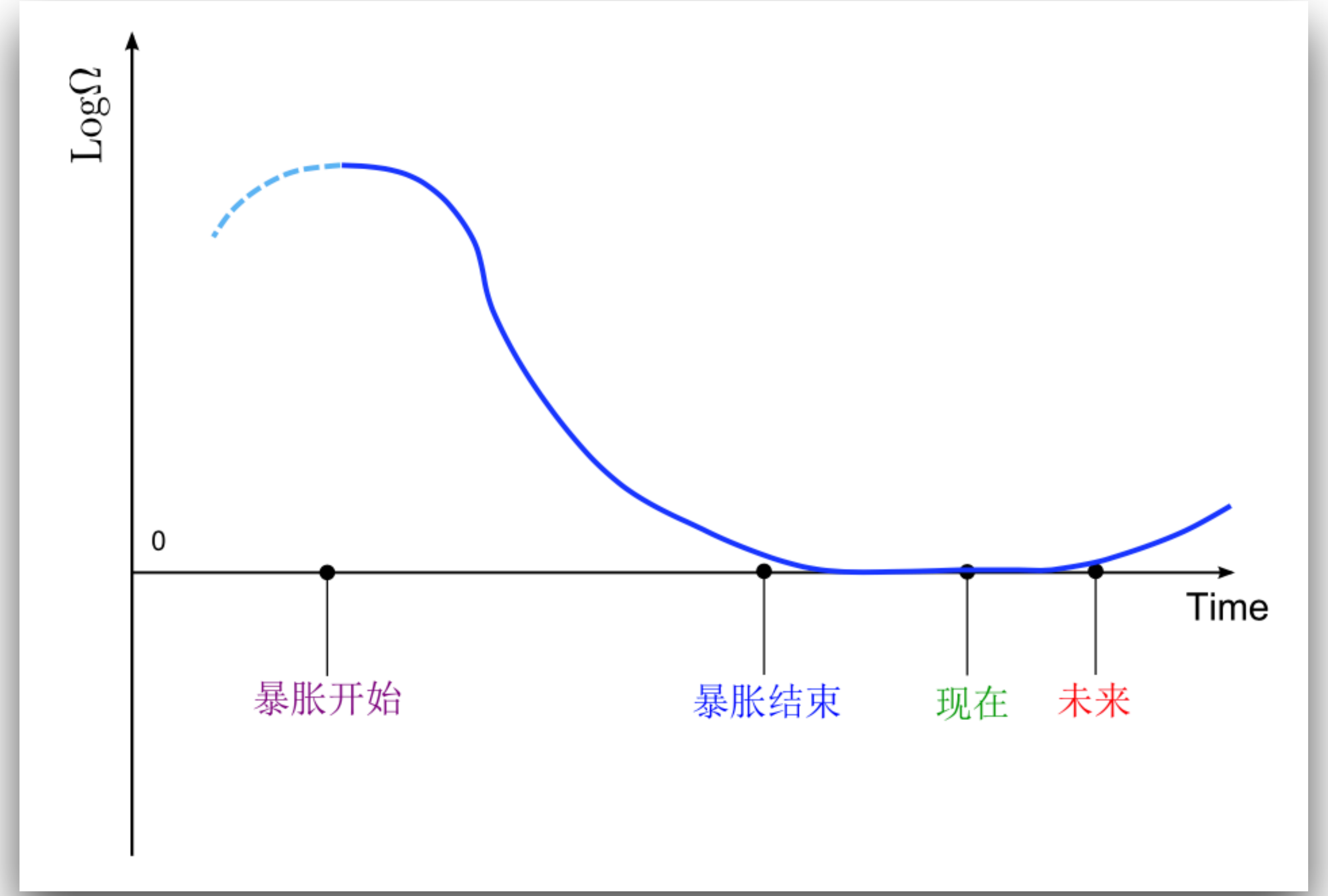
$$\frac{\rho_{pl}}{\rho_{eq}} = \left(\frac{E_{pl}}{E_{eq}}\right)^4 \sim 10^{108}$$

$H^2 \propto \rho \propto a^{-4}$

radiation era

radiation era covers most parts of the energy scale

$10^{54} \longleftarrow a^2 H^2 \propto \sqrt{\rho}$



monopole problem

GUT $\longrightarrow$ huge mount of stable magnetic monopole

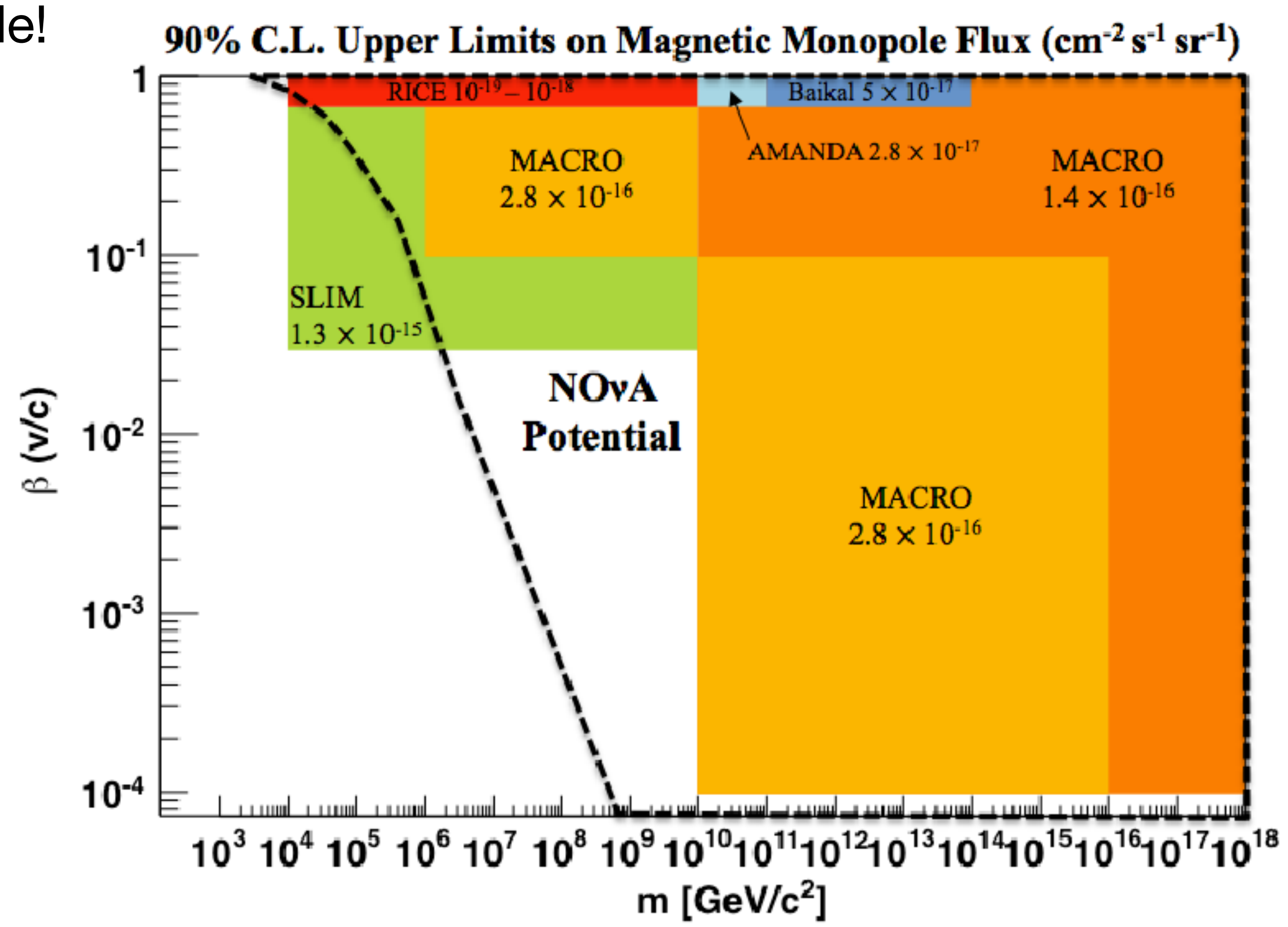
m~10^{16} GeV

$\rho_c \sim 10^{-29}[gm/cm^3]$

$\rho_{mon} > 10^{-18}[gm/cm^3]$

$\Omega = \rho_{mon}/\rho_c > 10^{11}$

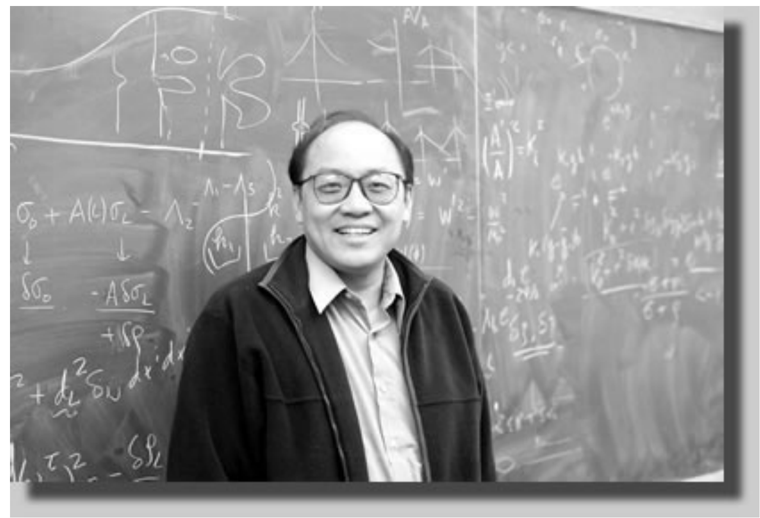
completely dominated by monopole!



The way out?

within 10^{-36} s, stretch the physical scale of the forward light-cone by a factor e^{60}

Guth 1980

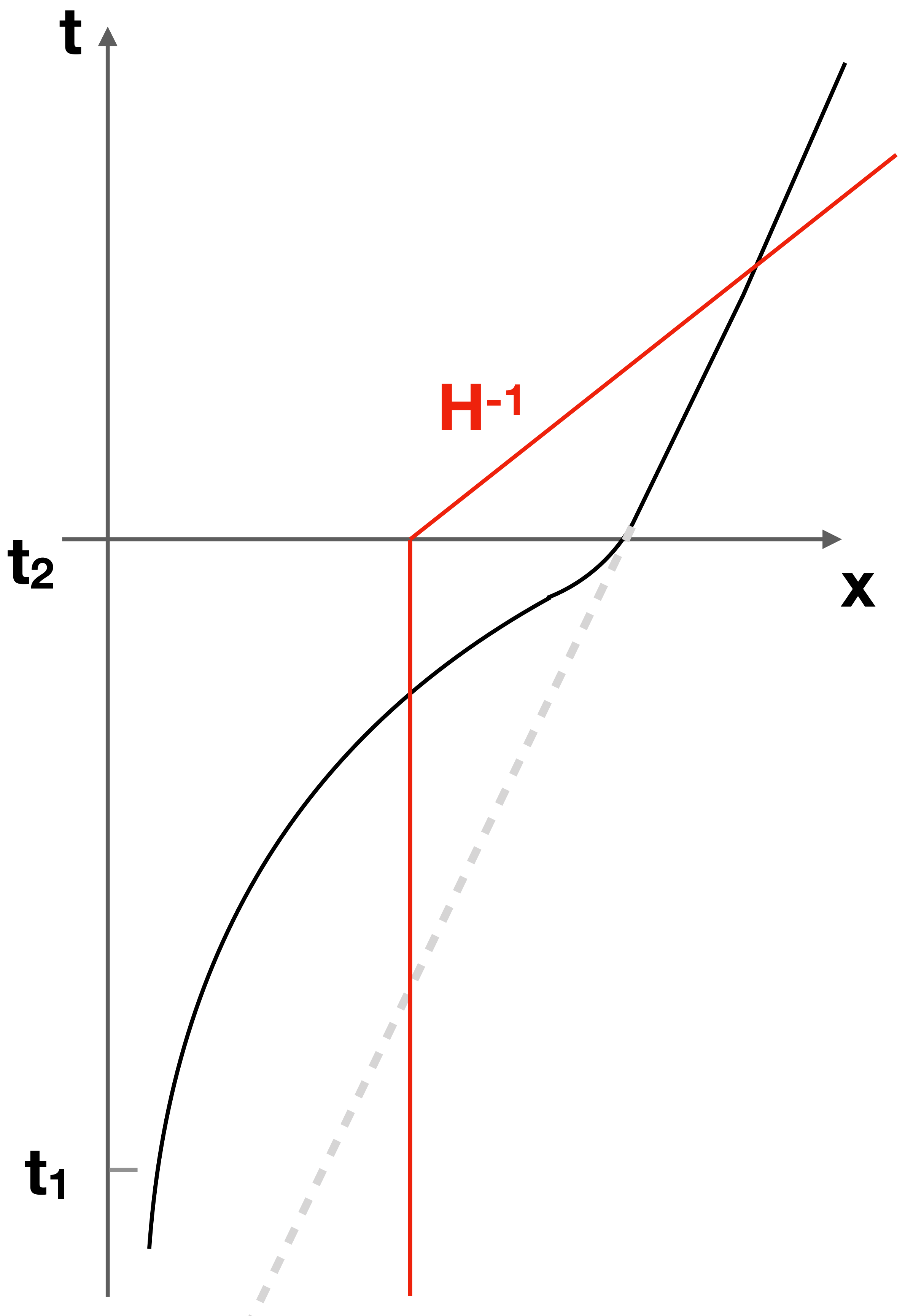


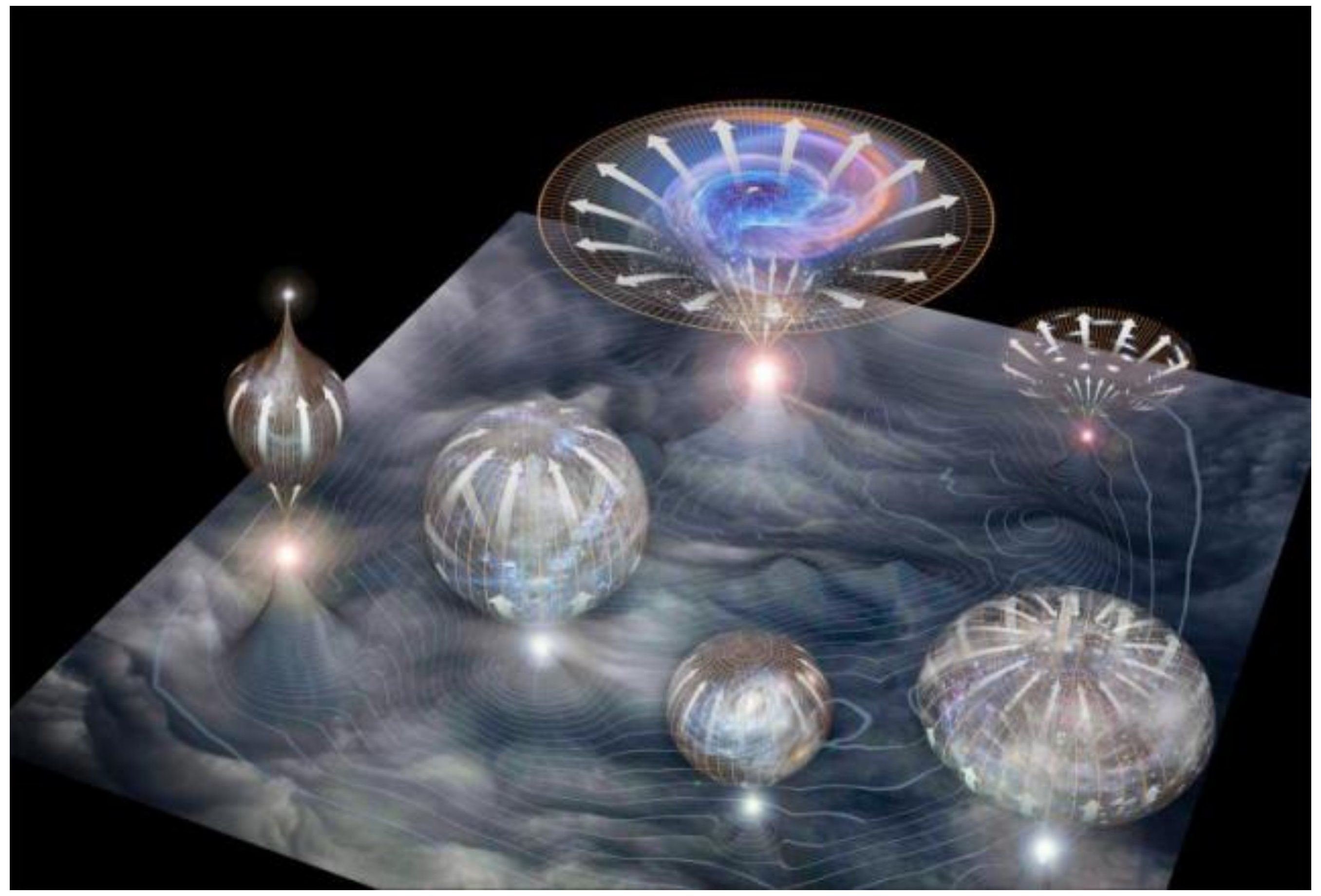
Henry Tye

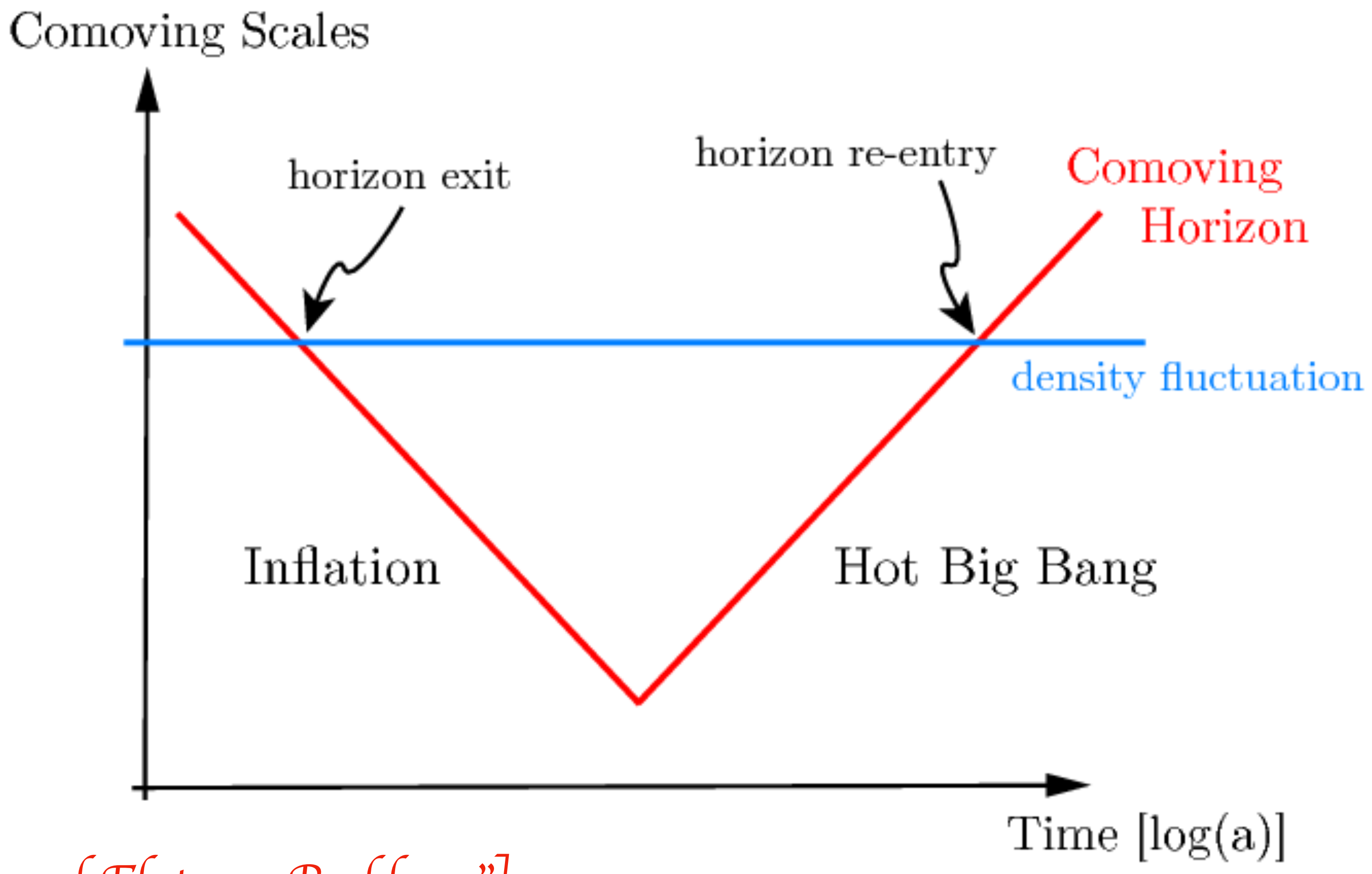
how to：qausi-de Sitter phase ⟶ exponential expansion

in RD/MD era, a~t# (power law), too slow!

$a = e^{H \cdot \Delta t}$ $H \cdot \Delta t = 60$ **H~ const**







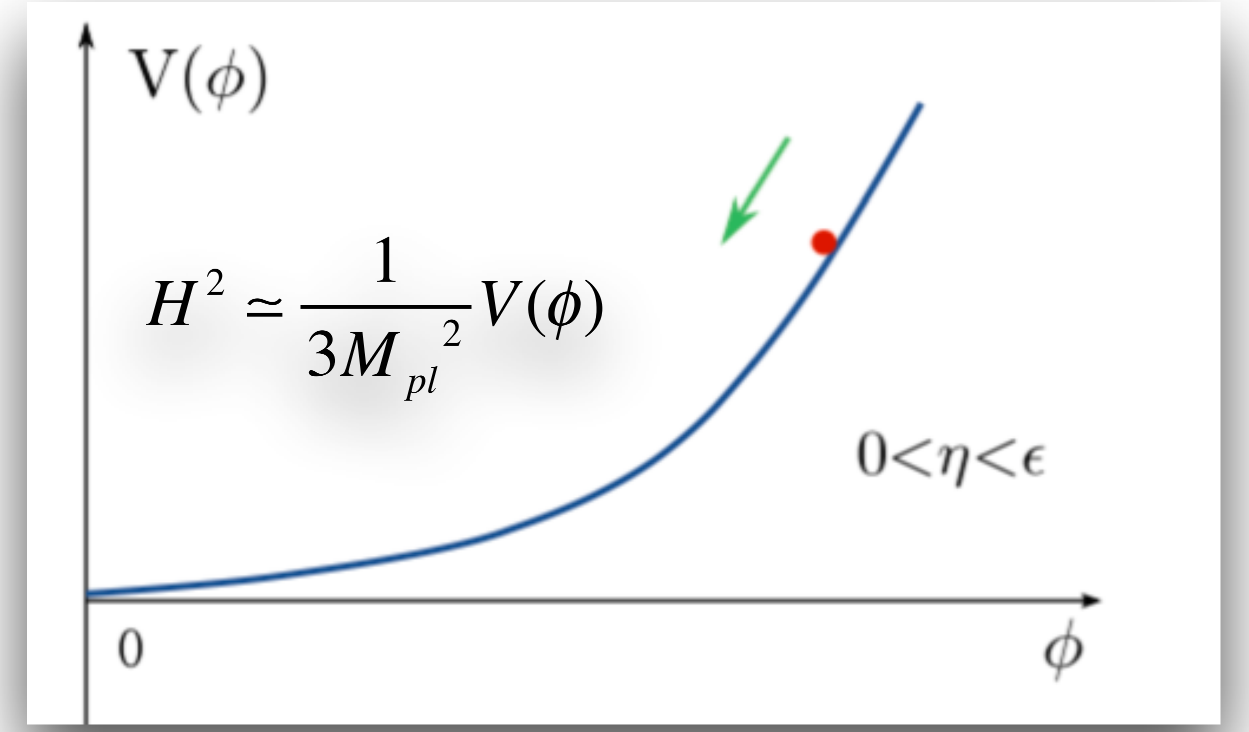
[Guth & Tye, 1979, PRL, "Phase Transitions and Magnetic Monopole Production in the Very Early Universe"]

[Guth, 1980, PRD, "The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems"]

mechanism: a scalar field slowly roll in its potential

$$S=\int d^4x\sqrt{-g}[\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-V(\phi)]$$

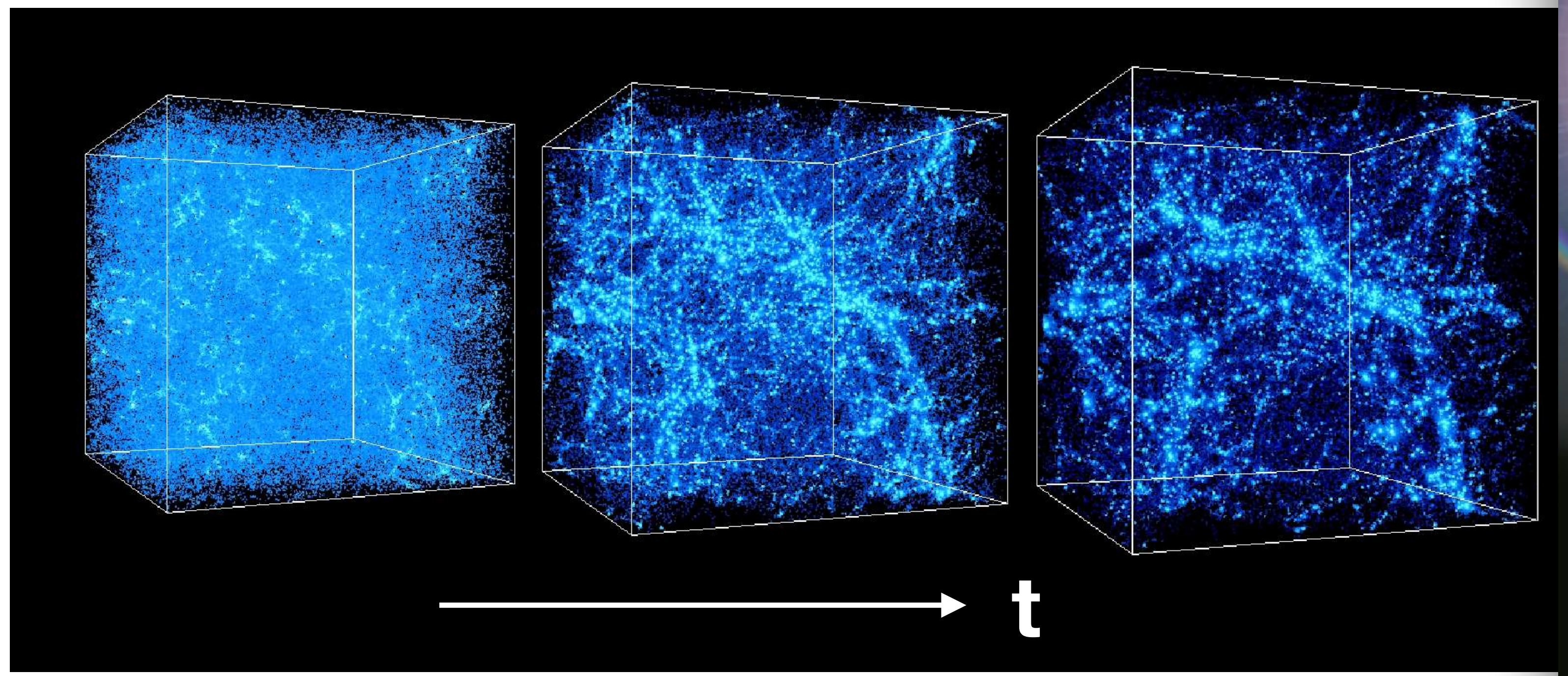
$$\rho=\frac{1}{2}\dot{\phi}^2+V(\phi) \qquad P=\frac{1}{2}\dot{\phi}^2-V(\phi) \qquad \dot{\phi}^2\ll V(\phi)\Leftrightarrow P\simeq-\rho \qquad H^2=\frac{1}{3M_{pl}^2}[\frac{1}{2}\dot{\phi}^2+V(\phi)]$$

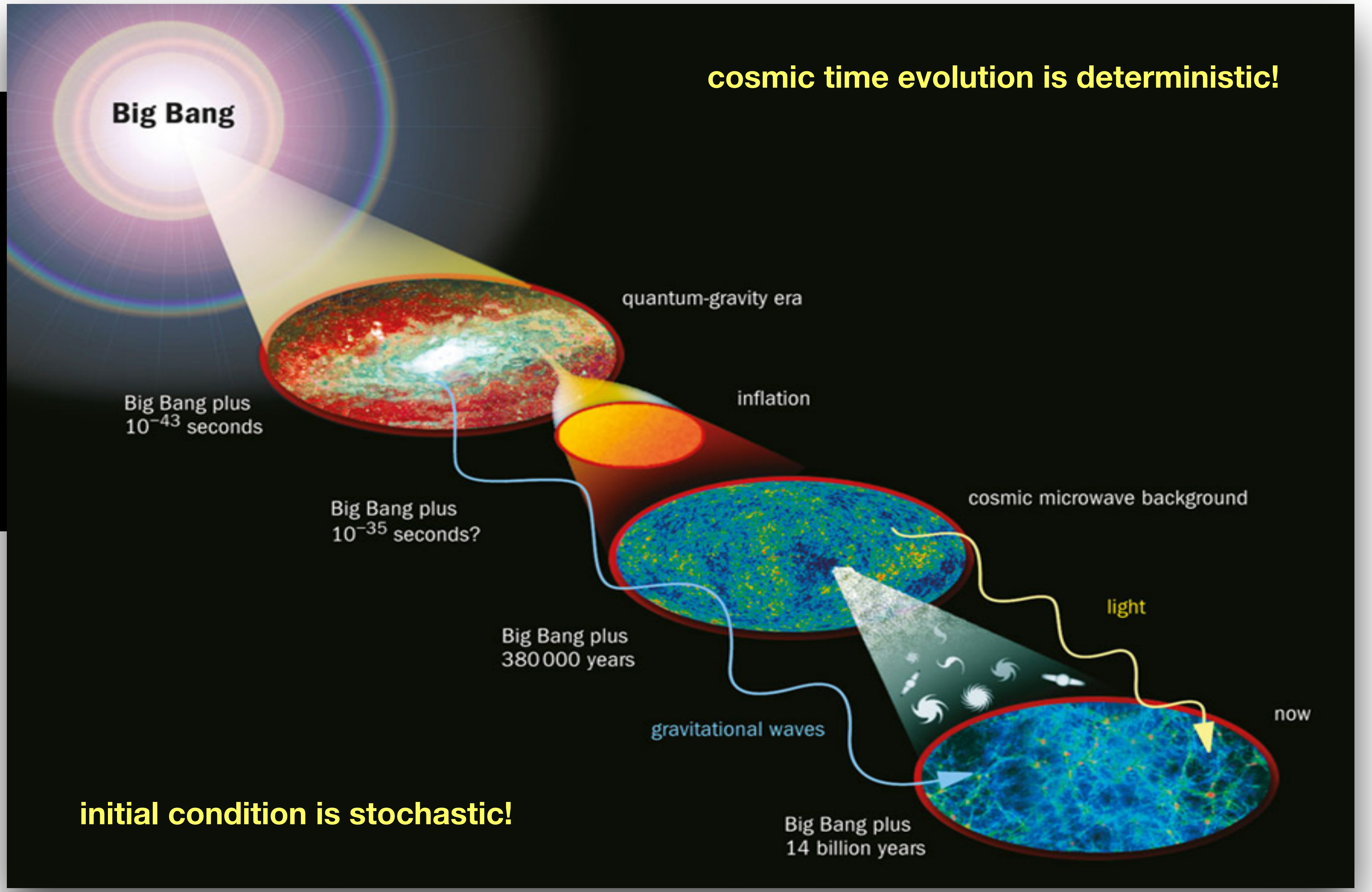


$$\epsilon = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2,$$
$$\eta = M_{pl}^2\left(\frac{V''}{V}\right),$$
$$\epsilon\ll 1\,, \qquad |\eta|\ll 1\,.$$

inflationary mechanism does not only solve several problems on the background level,

but also, naturally gives the initial conditions needed by the CMB and LSS formation! (we force on this)





$$P(k,z_0)=D^2(z_i,z_0)P_i(k)$$

obs evolution IC

inflaton action

$$S=\int \mathrm{d}\tau\,\mathrm{d}^3x\,\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)\right] \longrightarrow S=\int \mathrm{d}\tau\,\mathrm{d}^3x\left[\frac{1}{2}a^2\left((\phi')^2-(\nabla\phi)^2\right)-a^4V(\phi)\right]$$

plug unperturbed FRWL metric

$$\phi(\tau,\boldsymbol{x})=\bar\phi(\tau)+\frac{f(\tau,\boldsymbol{x})}{a(\tau)}$$

linear order action

$$S^{(1)}=\int \mathrm{d}\tau\,\mathrm{d}^3x\left[a\bar\phi' f'-a'\bar\phi' f-a^3V_{,\phi}f\right] \underset{\text{(deriv)}}{=} -\int \mathrm{d}\tau\,\mathrm{d}^3x\,a\left[\bar\phi''+2\mathcal{H}\bar\phi'+a^2V_{,\phi}\right]f$$

(deriv)

background field e.o.m

$$\bar\phi''+2\mathcal{H}\bar\phi'+a^2V_{,\phi}=0$$

quadratic action

$$S^{(2)}=\frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^3x\left[(f')^2-(\nabla f)^2-2\mathcal{H}ff'+\left(\mathcal{H}^2-a^2V_{,\phi\phi}\right)f^2\right] \underset{\text{(deriv)}}{=} \frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^3x\left[(f')^2-(\nabla f)^2+\left(\frac{a''}{a}-a^2V_{,\phi\phi}\right)f^2\right]$$

(deriv)

$$\boxed{S^{(2)}\approx\int \mathrm{d}\tau\,\mathrm{d}^3x\,\frac{1}{2}\left[(f')^2-(\nabla f)^2+\frac{a''}{a}f^2\right]}$$

$$\frac{V_{,\phi\phi}}{H^2}\approx\frac{3M_{\rm pl}^2V_{,\phi\phi}}{V}=3\eta_{\rm v}\ll 1 \qquad \frac{a''}{a}\approx 2a'H=2a^2H^2\gg a^2V_{,\phi\phi}$$

Mukhanov-Sasaki eq.

$$\boxed{f_{\boldsymbol{k}}''+\left(k^2-\frac{a''}{a}\right)f_{\boldsymbol{k}}=0}$$

'm_f^2'(negative mass sq)

sub-horizon limit

$$k^2\gg a''/a\approx 2\mathcal{H}^2 \qquad f_{\boldsymbol{k}}''+k^2f_{\boldsymbol{k}}\approx 0 \longrightarrow$$

Simple Harmonic oscillator with 0-mass
in Minkowski space (no feel of curvature)

$V_{\phi\phi}\propto m_f^{\ 2};m_f$ ~eta*H   in this energy level (M_{pl}>>H), inflaton behaves as massless particle

e.g.

$$V(\phi)=\frac{1}{2}m_f^{\ 2}\phi^2 \qquad H^2\simeq\frac{1}{3M_{pl}^{\ 2}}V(\phi)$$

$$\boxed{\bar\phi\sim M_{pl};\delta\phi\sim H}$$

validation of our calculation!

$$H\ll M_{pl}$$

we quantise $\delta\phi$ **NOT** $\bar\phi$

up to now,
no quantum gravity
theory available (@M_{pl} scale)

classical field

$$\boxed{f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = 0}$$

$$a(t) = e^{Ht} \qquad a(\tau) = \frac{\tau_0}{\tau} \quad \text{(deriv)} \qquad f_k'' + \left(k^2 - \frac{2}{\tau^2}\right) f_k = 0$$

general solution

$$f_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right)$$

For a classical vacuum, no reason to excite any state, so it is natural to choose $\alpha = \beta = 0$

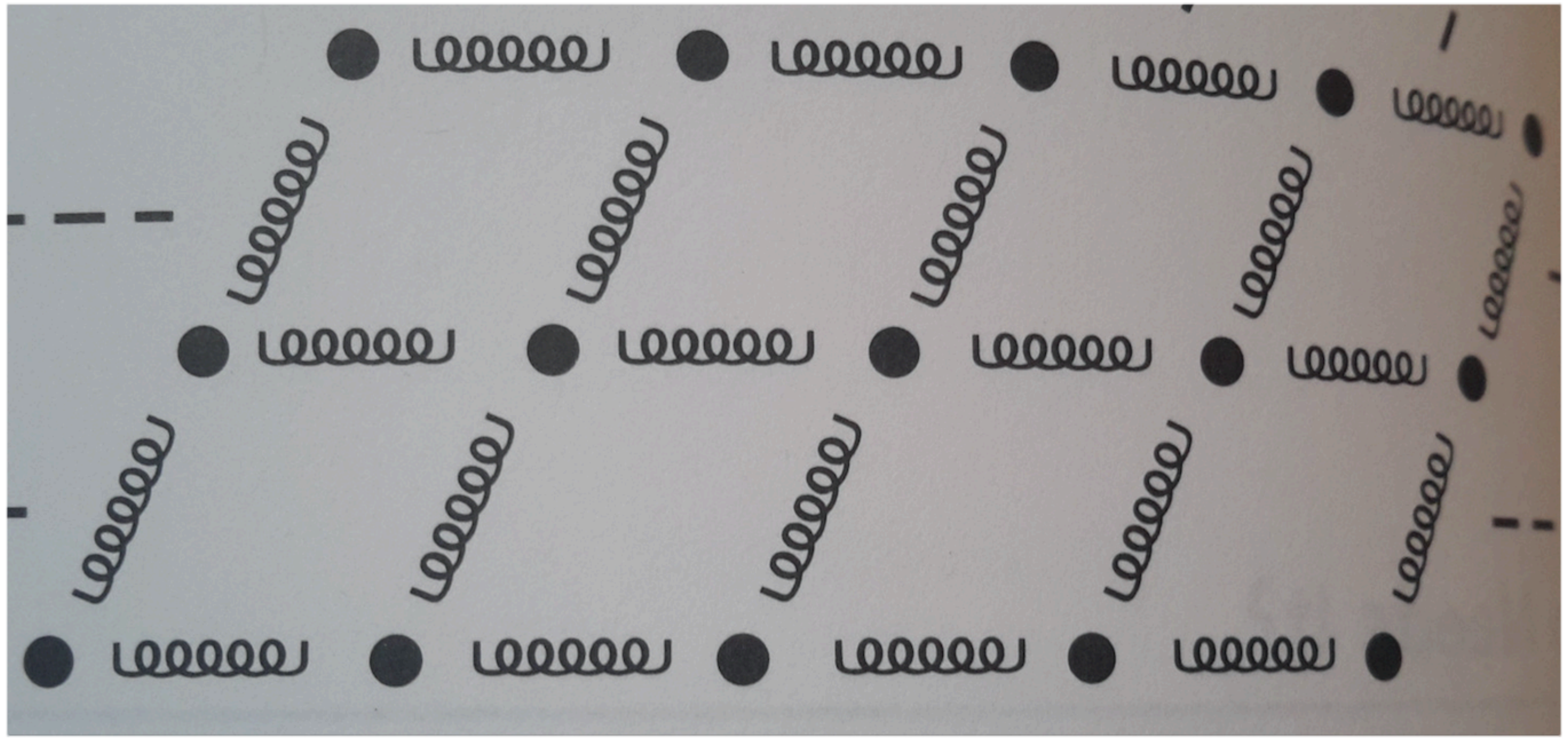
However, the **quantum fluct.** in the curved space-time, will naturally gives

$$\boxed{f_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)}$$

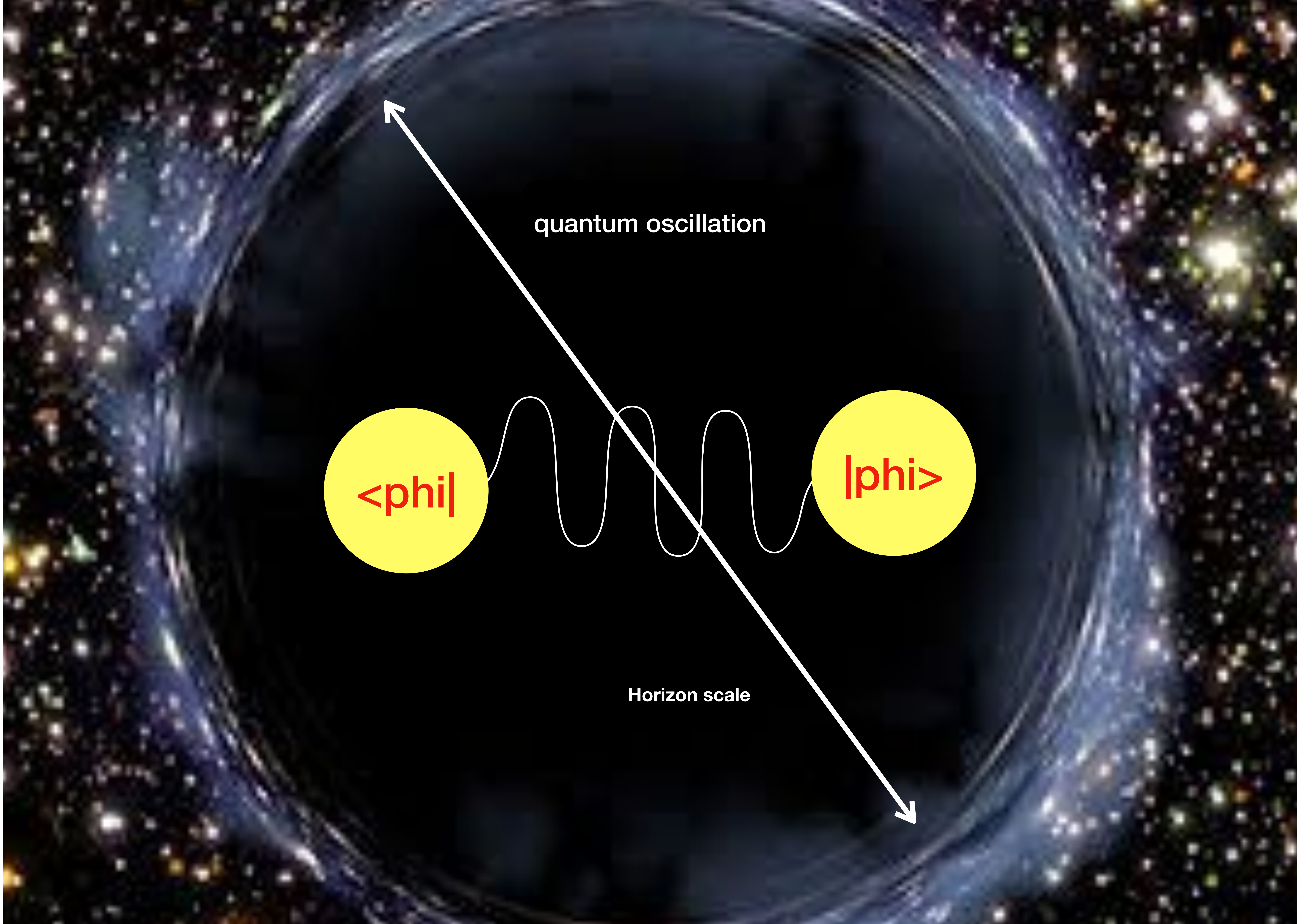
(Bunch-Davis vacuum)
(adiabatic state)
(no particle creation)

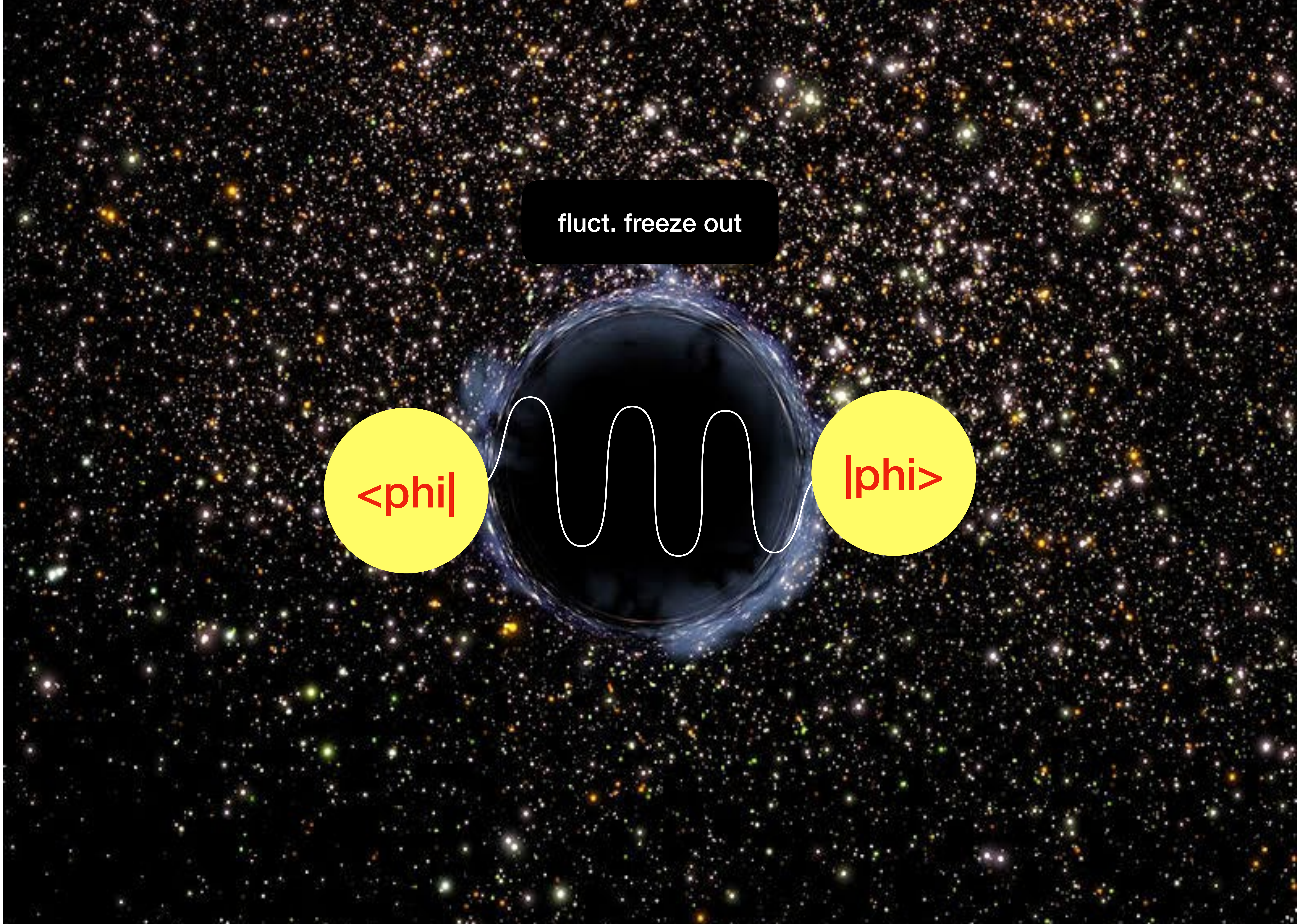
If we zoom in (time & space), a classical vacuum, is full of instantaneous particle creations and annihilations.

(off-set of the equilibrium position denotes for the particle creation/annihilation)



the quantum field view of space-time: string matrix


quantum oscillation
<phi|
|phi>
Horizon scale


fluct. freeze out
<phi|
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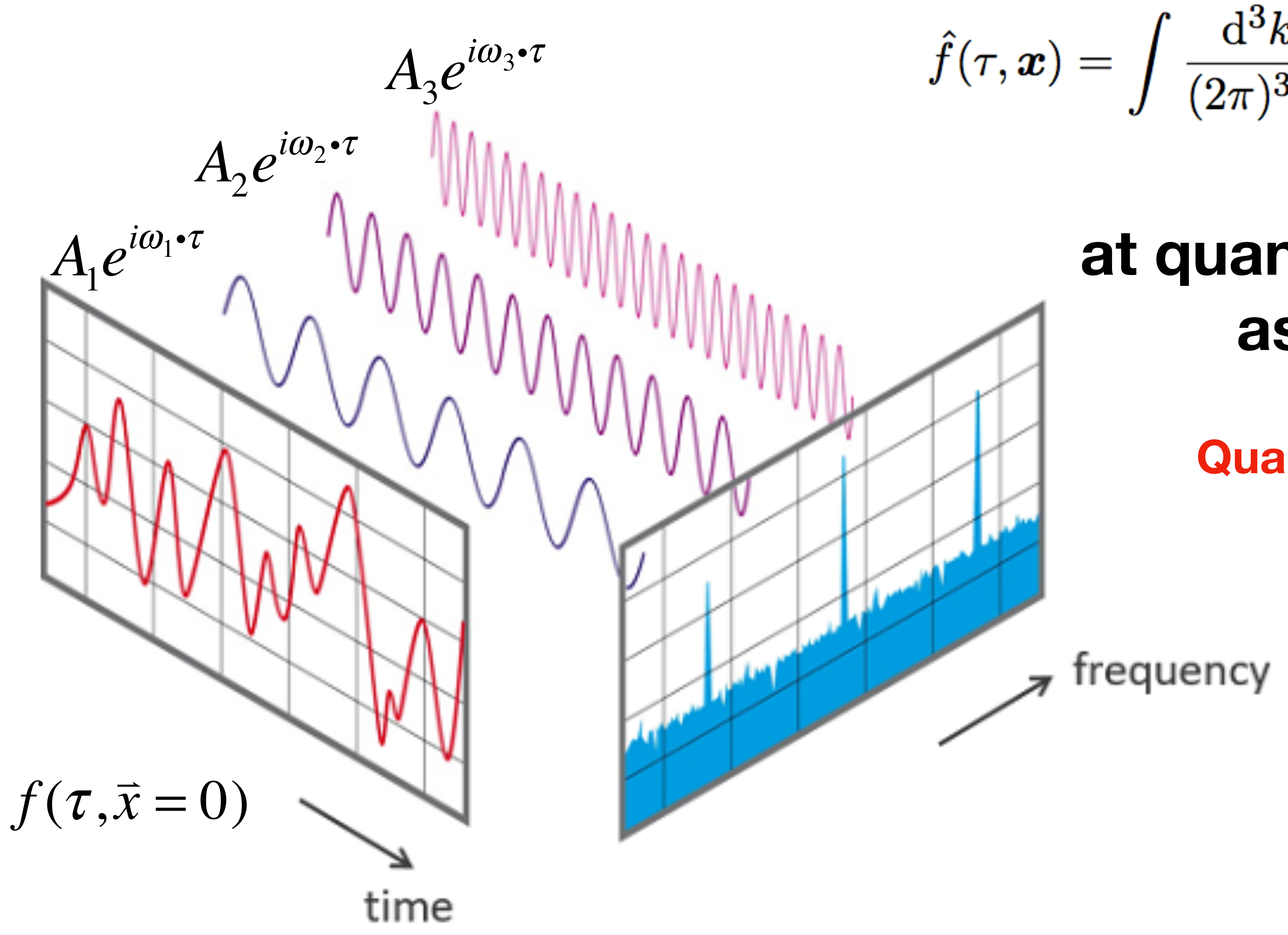
Let us fix a space point $\vec{x}=0$ **, record scalar field amplitude** $f(\tau,\vec{x}=0)$



$$\hat{f}(\tau,\boldsymbol{x})=\int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\left[f_k(\tau)\hat{a}_{\boldsymbol{k}}+f_k^*(\tau)a_{\boldsymbol{k}}^\dagger\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

at quantum level, the scalar field can be treated as an assembly of simple harmonics!

Quantum Field is a collection of Quantum mechanics

$$\langle\hat{f}\rangle=0 \qquad f(\tau,\vec{x})=\sqrt{<\hat{f}\bullet\hat{f}>}$$

classical solution quantum operator

$$\hat{f}=f\bullet\hat{\delta} \qquad <\hat{\delta}>=0,<\hat{\delta}\bullet\hat{\delta}>=1$$

Gaussian random variables

quantization of the pert.

$$\hat{f}(\tau, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left[f_k(\tau) \hat{a}_{\boldsymbol{k}} + f_k^*(\tau) a_{\boldsymbol{k}}^\dagger \right] e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\boxed{[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger] = \delta(\boldsymbol{k} + \boldsymbol{k}')}$$

$$\langle |\hat{f}|^2 \rangle \equiv \langle 0 | \hat{f}^\dagger(\tau, \boldsymbol{0}) \hat{f}(\tau, \boldsymbol{0}) | 0 \rangle$$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 k'}{(2\pi)^{3/2}} \langle 0 | \left(f_k^*(\tau) \hat{a}_{\boldsymbol{k}}^\dagger + f_k(\tau) \hat{a}_{\boldsymbol{k}} \right) \left(f_{k'}(\tau) \hat{a}_{\boldsymbol{k}'} + f_{k'}^*(\tau) \hat{a}_{\boldsymbol{k}'}^\dagger \right) | 0 \rangle$$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 k'}{(2\pi)^{3/2}} f_k(\tau) f_{k'}^*(\tau) \langle 0 | [\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger] | 0 \rangle$$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} |f_k(\tau)|^2 \hbar$$

$$= \int \mathrm{d} \ln k \, \frac{k^3}{2\pi^2} |f_k(\tau)|^2 \hbar$$ (deriv)

$$\langle \hat{f} \rangle = 0$$

mode function $f_k(\tau)$: is chosen to be the classical field solution

$$\boxed{f_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right)} \sqrt{\hbar}$$

conjugate momentum

$$[\hat{f}_{\vec{k}}(\tau), \hat{\pi}_{\vec{k}'}(\tau)] = i\delta(\vec{k} + \vec{k}')$$

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial f'} = f'$$

quantum effect

$$\boxed{f_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau} \right)}$$

The difference between classical & quantum pert.

for classical pert. (α, β) could be **arbitrary** large

for quantum pert. the wave function must be **unitary** (probability normalised to unity) $\alpha^2 + \beta^2 = 1$

---

decoherence

two quantum states separated by a scale k^{-1}, are in coherence! (correlated amplitude and phase)

However, the afterward cosmic evolution is classical process, e.g. galaxy formation

quantum $\xrightarrow[\text{decoherence}]{}$ **classical**

sub-horizon

$$f_k \sim \frac{e^{-ik\tau}}{\sqrt{2k}} \qquad \pi_k \sim -\frac{ike^{-ik\tau}}{\sqrt{2k}}$$

$$< 0 | [\hat{f}_k, \hat{\pi}_{k'}] | 0 >= i\delta(k + k')$$ (deriv)

non-commute $\longrightarrow$ quantum state

super-horizon

$$f_k \sim -\frac{i}{\sqrt{2} k^{3/2} \tau} \qquad \pi_k \sim \frac{i}{\sqrt{2} k^{3/2} \tau^2}$$

$$< 0 | [\hat{f}_k, \hat{\pi}_k] | 0 >= 0$$ (deriv)

commute $\longrightarrow$ classical state

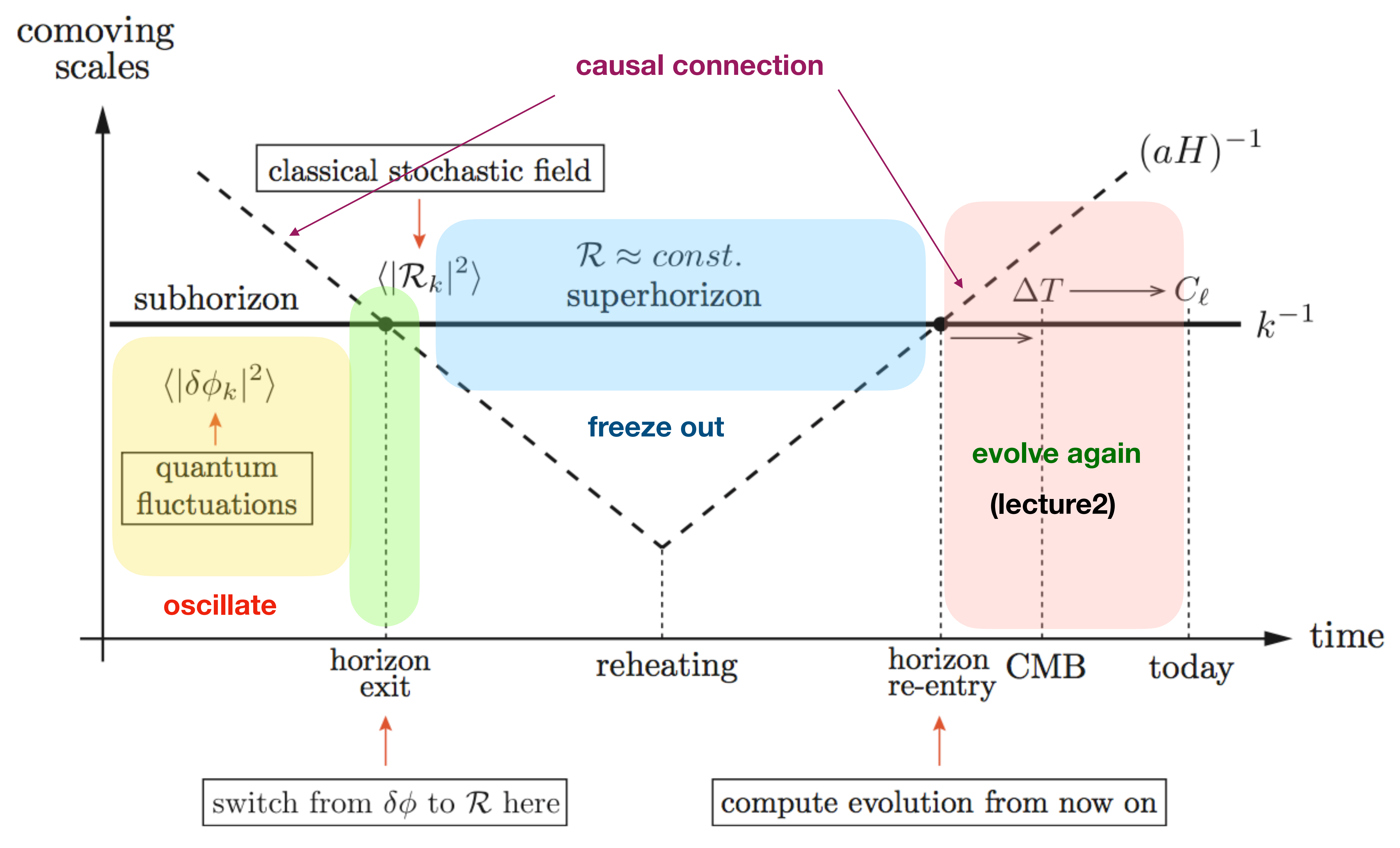
primordial scalar power spectrum

$a(\tau)=\frac{\tau_0}{\tau}$ $aH=\mathcal{H}$ $a=-1/H\tau$ (deriv)

$$\langle|\hat{f}|^2\rangle = \int \mathrm{d}\ln k\, \frac{k^3}{2\pi^2}|f_k(\tau)|^2$$

dimensionless power spectrum $\boxed{\Delta_f^2(k,\tau)\equiv\frac{k^3}{2\pi^2}|f_k(\tau)|^2}$

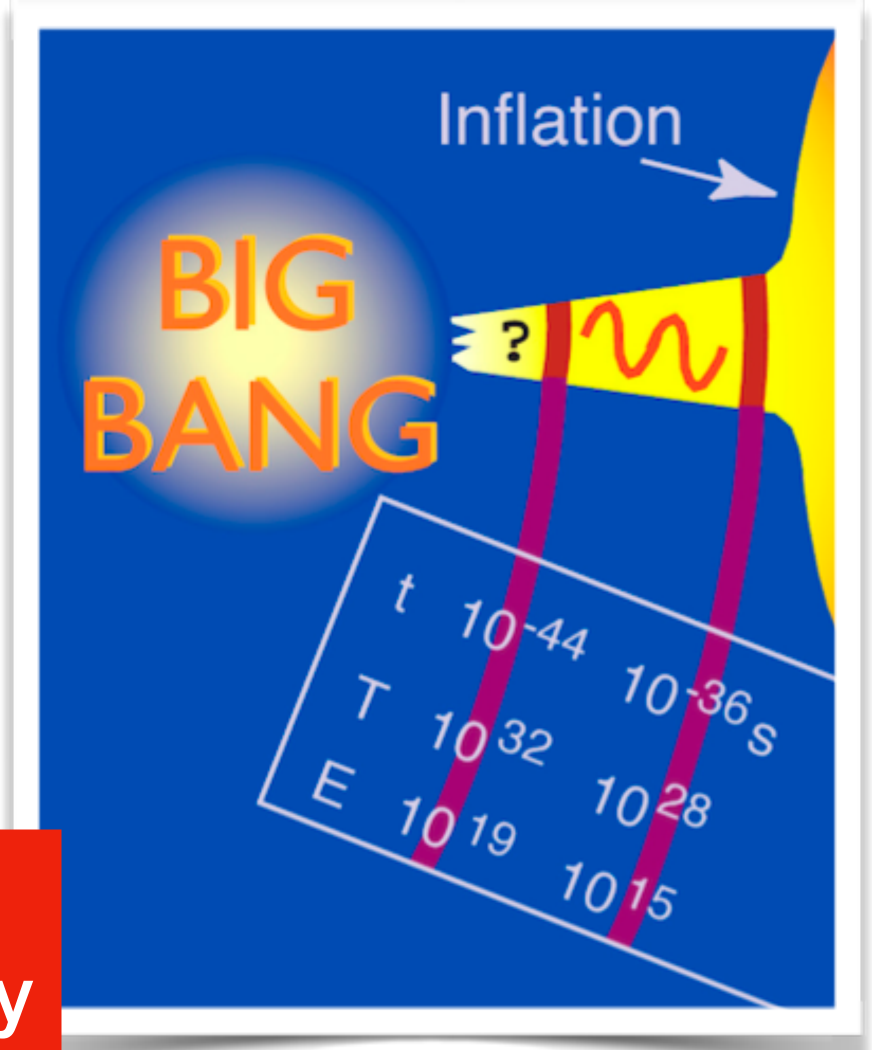
super-horizon mode $f_k\sim-\frac{i}{\sqrt{2k^{3/2}}\tau}$



$$\Delta^2_{\delta\phi}(k,\tau)=a^{-2}\Delta_f^2(k,\tau)=\left(\frac{H}{2\pi}\right)^2$$ (deriv)

the amplitude of the pert. is proportional to inflationary energy scale!

(by measuring the amp we can 'know' the inflation energy scale)



[Pb2.]

$$\Delta^2_{\mathcal{R}}=\frac{1}{2\varepsilon}\frac{\Delta^2_{\delta\phi}}{M^2_{\rm pl}}\,,\qquad \text{where}\quad \varepsilon=\frac{\frac{1}{2}\dot{\phi}^2}{M^2_{\rm pl}H^2}$$

gauge-inv curvature pert.

$$\boxed{\Delta^2_{\mathcal{R}}(k)=\frac{1}{8\pi^2}\frac{1}{\varepsilon}\frac{H^2}{M^2_{\rm pl}}\bigg|_{k=aH}}\quad \textbf{or}\quad \boxed{\Delta^2_{\mathcal{R}}=\frac{1}{12\pi^2}\frac{V^3}{M^6_{\rm pl}(V')^2}}$$

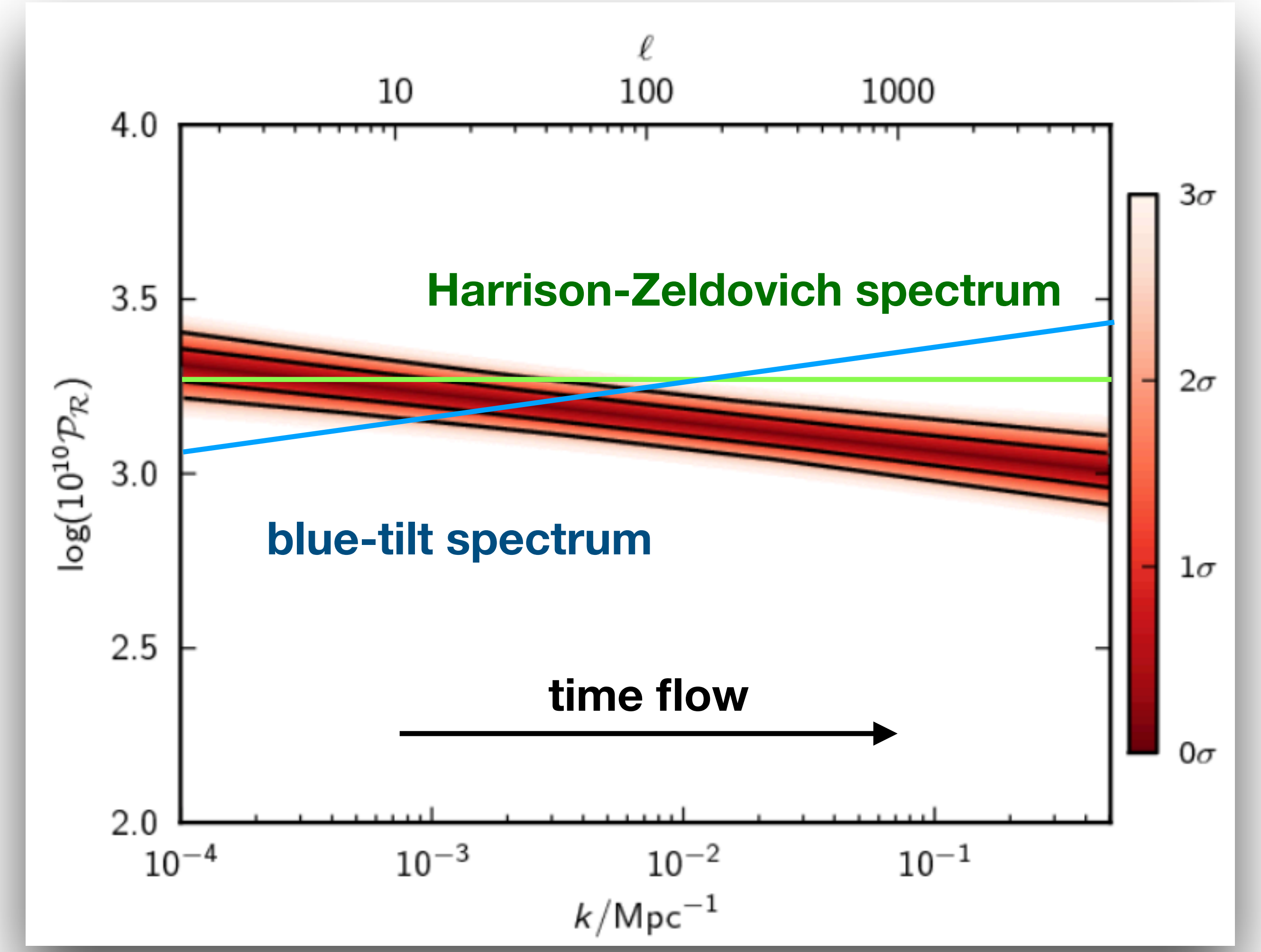
$H^2\propto V$ $\Delta_R\sim(V,V')$

scalar pert. per. se. **could NOT determine the inflation energy scale!** (its amp also depends **on the potential slop)**

nearly scale-inv power spectrum

$$\Delta^2_{\mathcal{R}}(k) = \frac{1}{8\pi^2}\frac{1}{\varepsilon}\frac{H^2}{M^2_{\rm pl}}\bigg|_{k=aH}$$

if ε, H purely constant $\longrightarrow$ exact scale-inv



$$\varepsilon \equiv -\frac{\dot H}{H^2} \qquad \eta \equiv \frac{d\log\varepsilon}{dN} \qquad a = e^N = e^{\int H\,dt}$$

1st time derivative

2nd time derivative

$$\Delta^2_{\mathcal{R}}(k) \equiv A_s\left(\frac{k}{k_\star}\right)^{n_s-1}$$

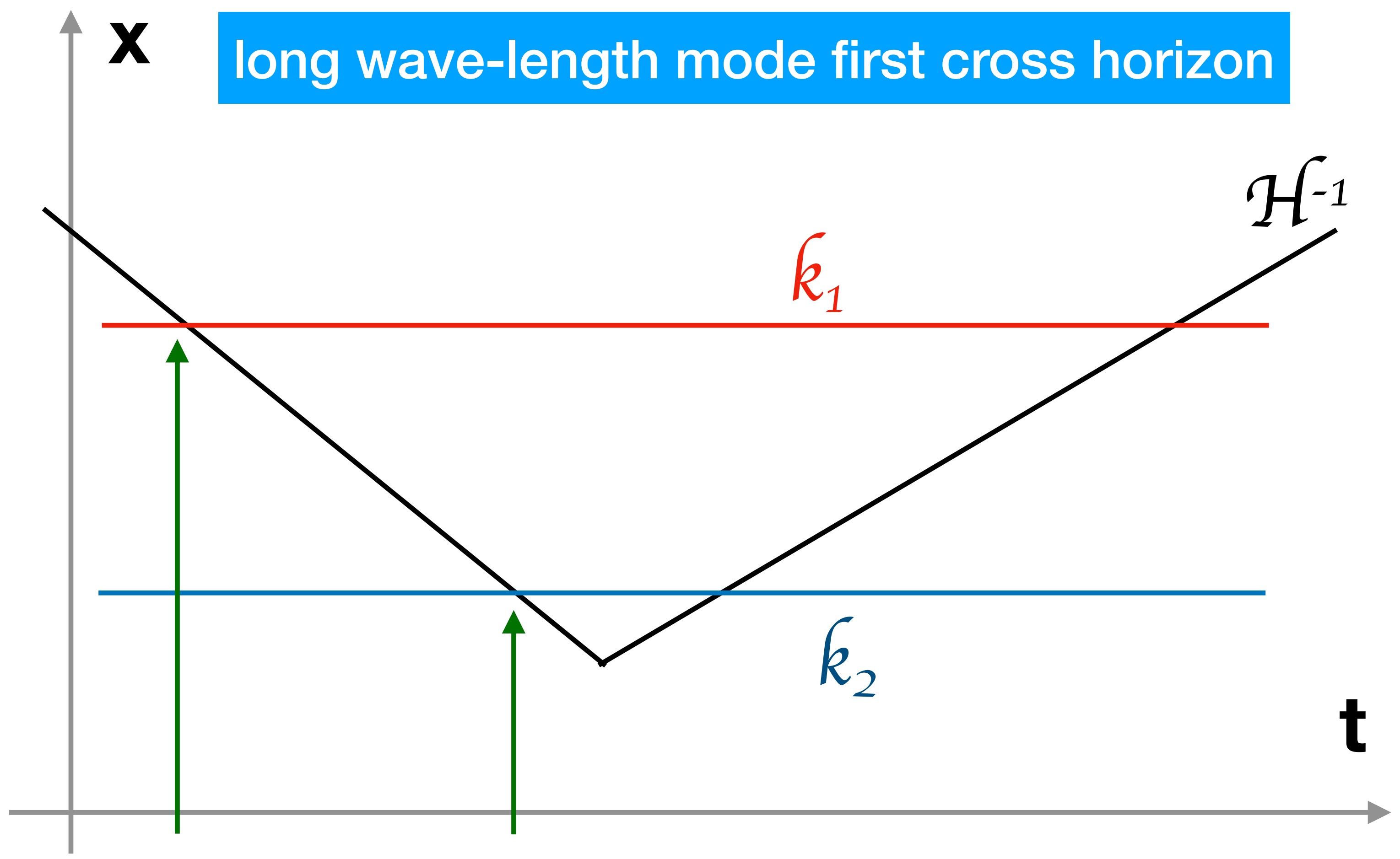
$$n_s - 1 = \frac{d\log\Delta_R^2}{d\log k} \sim -2\varepsilon - \eta$$

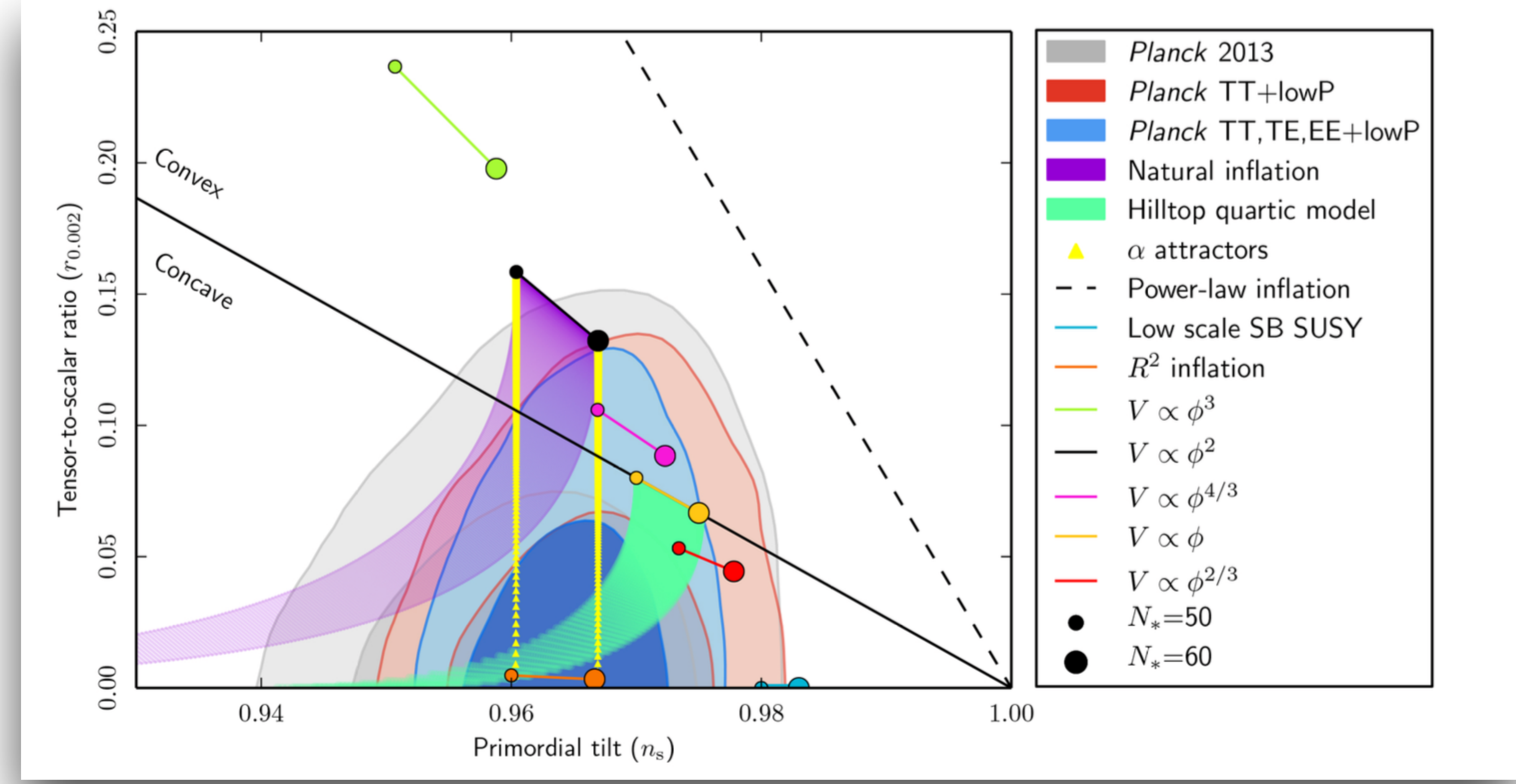
(deriv)

$$A_s = (2.196 \pm 0.060)\times 10^{-9}$$

$$n_s = 0.9603 \pm 0.0073$$

- **red-tilt:** $n_s - 1 < 0$ amp is large on the large scale
- **blue-tilt:** $n_s - 1 > 0$ amp is large on the small scale





tensor pert. (primordial gravitational waves)

@ such high energy scale, if inflaton could have instantaneous particle creation/annihilation, why not the graviton?

no symmetry prevent this!

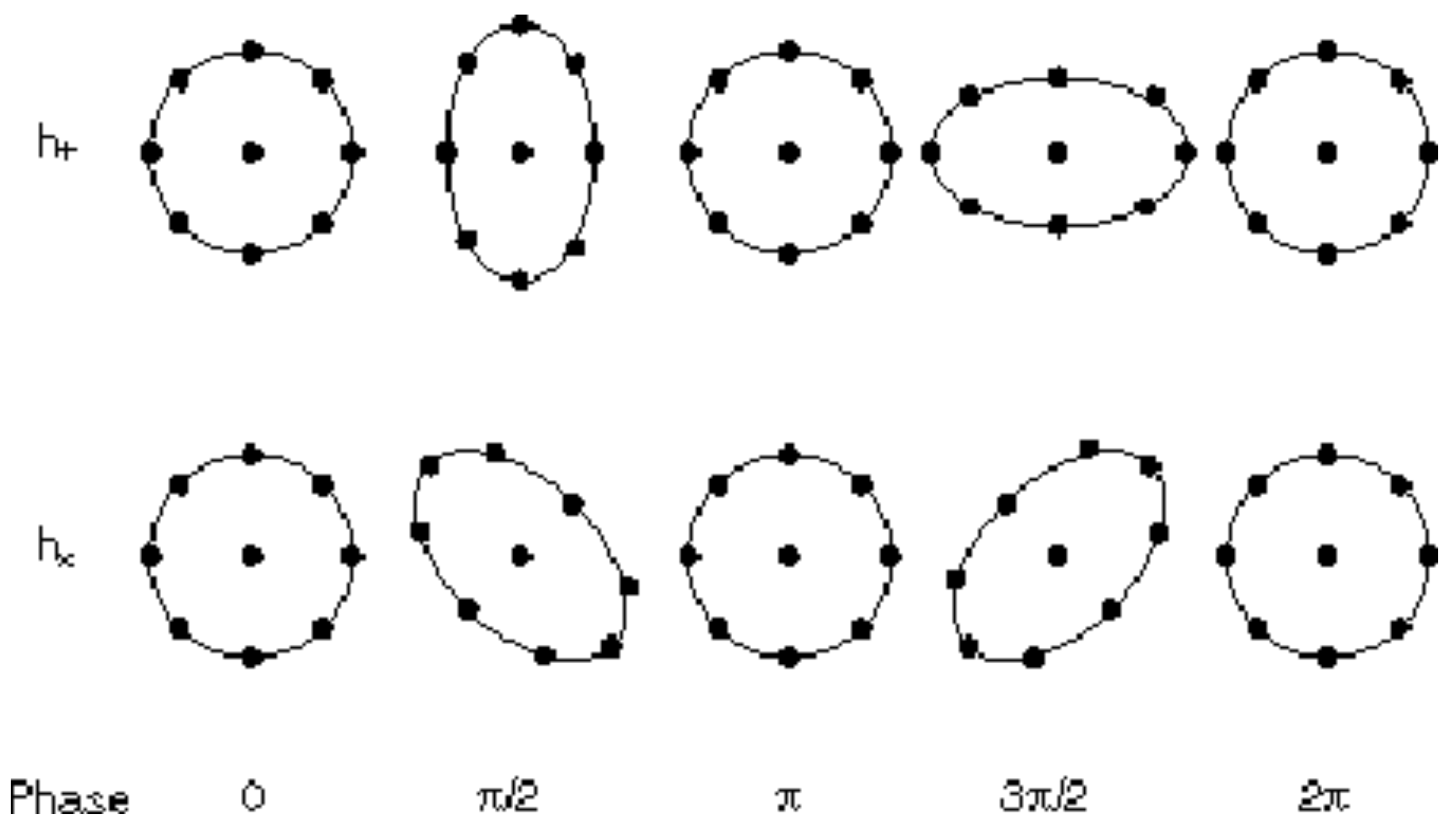
$$ds^2 = a^2(\tau)\left[d\tau^2 - (\delta_{ij} + 2\hat{E}_{ij})dx^i dx^j\right]$$

$$\frac{M_{\rm pl}}{2}\, a\hat{E}_{ij} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} f_+ & f_\times & 0 \\ f_\times & -f_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S = \frac{M_{\rm pl}^2}{2}\int d^4x\sqrt{-g}\,R \quad \Rightarrow \quad S^{(2)} = \frac{M_{\rm pl}^2}{8}\int d\tau\, d^3x\, a^2\left[(\hat{E}_{ij}')^2 - (\nabla \hat{E}_{ij})^2\right]$$

[Pb3.]

$$S^{(2)} = \frac{1}{2}\sum_{I=+,\times}\int d\tau d^3x\left[(f_I')^2 - (\nabla f_I)^2 + \frac{a''}{a}f_I^2\right]$$



exactly the same as scalar pert.

[Pb4.]

$$\boxed{\Delta_t^2(k) = \frac{2}{\pi^2}\frac{H^2}{M_{\rm pl}^2}\bigg|_{k=aH}} \quad \textbf{V.S.} \quad \boxed{\Delta_{\mathcal{R}}^2(k) = \frac{1}{8\pi^2}\frac{1}{\varepsilon}\frac{H^2}{M_{\rm pl}^2}\bigg|_{k=aH}}$$

only depends on H!

direct probe of inflation scale! that is why we need measure PGW! **fundamental physics**

$$\Delta_t^2(k) \equiv A_t\left(\frac{k}{k_\star}\right)^{n_t} \qquad r \equiv \frac{A_t}{A_s}$$

(see pic in prev)

Exercise.—Show that

[Pb5.]

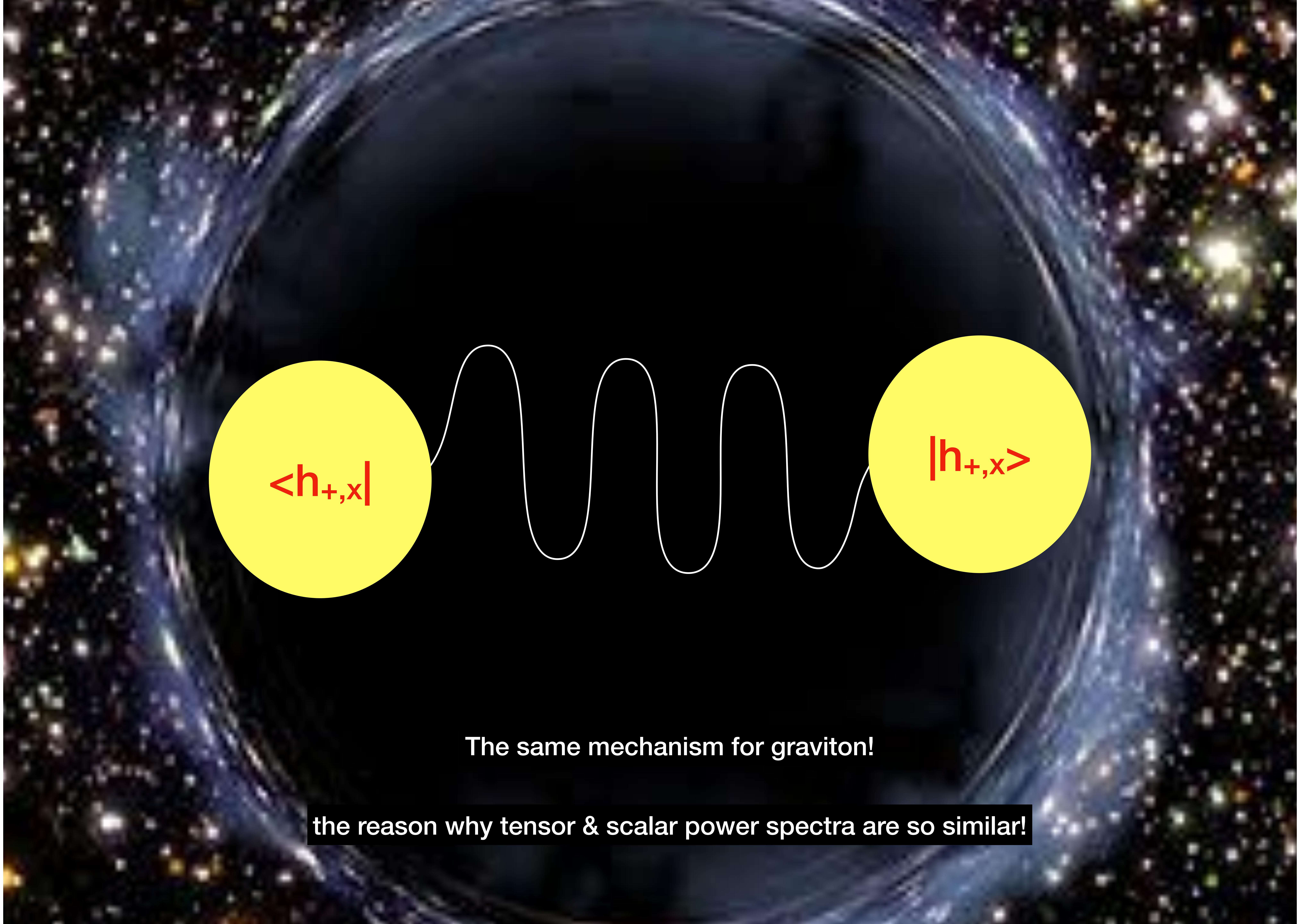
$$r = 16\varepsilon$$
$$n_t = -2\varepsilon\ .$$

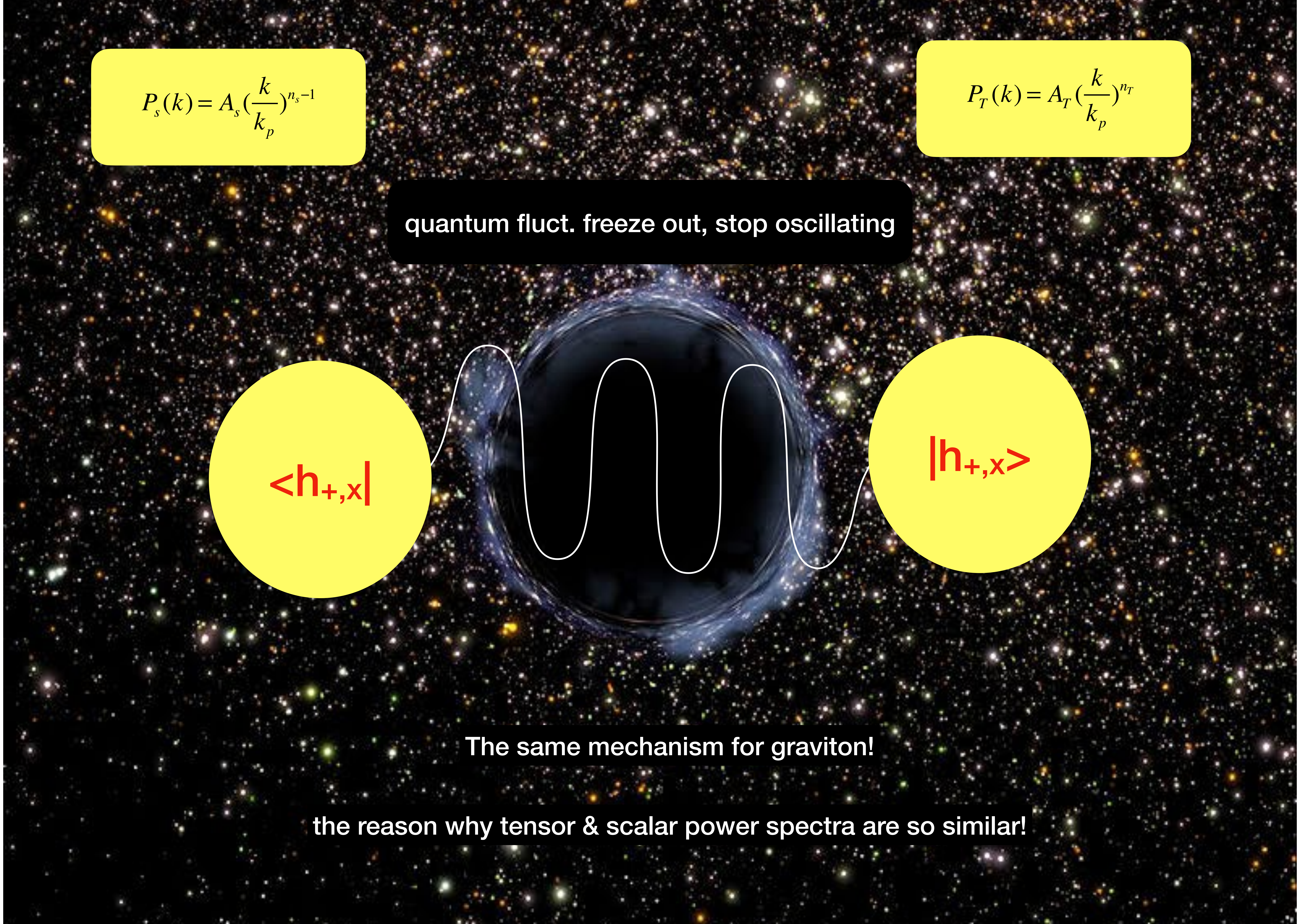
Notice that this implies the consistency relation $n_t = -r/8$.

scalar spec can be both red & blue

tensor spec must be both blue!

(otherwise, violate null energy condition)


$\langle h_{+,x}|$
$|h_{+,x}\rangle$
The same mechanism for graviton!
the reason why tensor & scalar power spectra are so similar!


$P_s(k) = A_s(\frac{k}{k_p})^{n_s-1}$
$P_T(k) = A_T(\frac{k}{k_p})^{n_T}$
quantum fluct. freeze out, stop oscillating
$\langle h_{+,x}|$
$|h_{+,x}\rangle$
The same mechanism for graviton!
the reason why tensor & scalar power spectra are so similar!

Further reading

- **Baumann lecture note/**Chapter 6
- **Physical Foundations of Cosmology/**Mukhanov